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THE

# ELEMENTS OF ALGEBRA,

WITH NUMEROUS EXAMPLES,

For the Use of Schools.

By REV. ROBERT FOWLER, M.A.,
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LONDON:
SIMPKIN, MARSHALL, AND CO.
1861.

~~180. e. 66~~
181. c. 150.

*Shortly will be published, separately,*

THE ANSWERS TO THE EXAMPLES.

BIBLIOTHECA·BODLEIANA·

LONDON: PRINTED BY W. CLOWES AND SONS, STAMFORD STREET,
AND CHARING CROSS.

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# PREFACE.

My apology for writing the following pages is that I have taught more or less of Algebra to above five hundred pupils; and my motive has been to supply some of those aids to beginners which a twelve years' experience has shown me to be necessary, but which are not to be found in the works in Algebra with which I am acquainted.

Three of the main difficulties in the way of beginners are the following:—

1. The want of model examples to guide them in working out their solutions in a neat and methodical form.

2. The want of experience and dexterity in resolving algebraical expressions into their simple factors.

3. Indistinct conceptions of the meanings of the symbols employed in general formulæ.

I have therefore placed models of the working out of questions at the head of each chapter of examples.

A chapter is devoted to the resolution of algebraical expressions into their factors; and I would recommend all teachers who use this book to put their pupils through this chapter before they begin fractions.

I have also enlarged upon and illustrated the meaning of the symbols in the general formulæ wherever they occur in the work.

In the chapter on Quadratic Equations several examples are given to explain the meanings of the two values of the unknown quantity, and a general rule given in page 168 to obtain the interpretation of the second value.

In the proof of the Binomial Theorem for positive integers, the appeal to the principle of induction has not been made, because it seems to me to be unnecessary. I have also discarded Euler's proof for fractional indices, because I have found very few beginners in Algebra who could make anything of it.

Very few examples of multiplication and division with fractional or negative indices are given in the early part; but the examples in page 116, &c., may be used for exercise in them.

R. F.

# A L G E B R A.

## CHAPTER I.

### DEFINITIONS.

**1.** IN addition to the *figures* used in Arithmetic, the *letters* of the Alphabet, $a$, $b$, $c$, $x$, $y$, $z$, &c., are used in Algebra to denote numbers and quantities. Thus we may speak of a line $a$ feet in length, of a stone weighing $x$ lbs.; also of the product of two quantities $a$ and $b$, &c.

As examples of the difference in Arithmetical and Algebraical notation, take the following:

(1.) If 2 lbs. of sugar cost 14*d.*, what will 7 lbs. cost?

By the Rules of Proportion we know that the price of 7 lbs. will be

$$\frac{7\times 14}{2}=49d.=4s.\ 1d.$$

(2.) If $a$ lbs. of sugar cost $p$ pence, what will $b$ lbs. cost?

Here $a$ represents the given number of lbs.

$b$ represents the number of lbs. whose price is required.

$p$ represents the given number of pence.

And question (2) is of the same nature as (1); therefore the price of $b$ lbs. will be

$$\frac{b\times p}{a}\text{ pence.}$$

(1) is an Arithmetical question.

(2) is an Algebraical question.

**2.** $+$ (read *plus*, more) signifies that the quantity to which it is prefixed is to be added, when possible.

$-$ (read *minus*, less) signifies that the quantity to which it is prefixed is to be subtracted, when possible.

Thus, $5+3$ means that 3 is to be added to 5;

and $5-3$ means that 3 is to be subtracted from 5.

So, $a+b$ means that $b$ is to be added to $a$; $a-b$ means that $b$ is to be subtracted from $a$.

When no sign is prefixed to the letter at the beginning of an expression, $+$ is understood.

Thus, $a+b$ means $+a+b$; or, $a$ and $b$ are both to be added.

Quantities with the *plus* sign before them are called *positive:* those with the *minus* sign are called *negative.*

**3.** The sign × (read *into*) is the sign of Multiplication.

Thus, $4 \times 3$ means 4 multiplied into 3.

$a \times b \times x \times y$ means the product of the four quantities $a$, $b$, $x$, and $y$.

Sometimes a dot (.) is used instead of the sign ×.

Thus, $a.b.x.y$ means the same as $a \times b \times x \times y$.

When letters are written in a row without any sign between them, the sign × is understood.

Thus, $a\ b\ x\ y$ means the same as $a \times b \times x \times y$.

**4.** The sign ÷ (read *by*) is the sign of Division.

Thus, $12 \div 4$ means that 12 is to be divided by 4;

Also $\frac{a}{b}$ and $a \div b$ mean that $a$ is to be divided by $b$.

**5.** The sign = (read *equal to*) is the sign of equality.

Thus, $5x = 3a + 4$ means that five times $x$ are equal to three times $a$ with four added.

**6.** The sign > means *greater than;*
The sign < means *less than.*

Thus, $a > 2b$ means that $a$ is greater than twice $b$.

**7.** The sign ∴ means *therefore;*
and ∵ means *because.*

**8.** The sign ( ) is called a *bracket.* It is used to enclose several separate quantities in one expression.

Thus, $(a + 2b - 3c)$ is a bracketed expression.

If any quantity or sign be written outside the bracket, each of the quantities within the bracket is equally affected by it.

**9.** When a quantity consists of two or more quantities multiplied together, each of the separate quantities is called a *factor*, and the whole quantity is called the *product* of the separate factors.

Thus, $3abc$ is the product of the four factors, $3$, $a$, $b$, $c$.

**10.** When all the factors of a product are equal, the product is said to be a *power* of the factor. The order of the power is the same as the number of the factors.

Thus, $5 \times 5$ or 25 is the second power of 5.
$5 \times 5 \times 5$ or 125 is the third power of 5.
Also, $a \times a \times a \times a$ or $a.a.a.a$ is the fourth power of $a$.

**11.** Instead of writing all the factors of a power as in the last article, the power is represented by writing the factor once, and denoting, by a small figure at the top, the number of times the factor is repeated.

Thus, $a.a.a.a$ is represented by $a^4$.
$a.a.a$ „ $a^3$.

The small figure that denotes the power to which the quantity is raised is called the *index* of the power, or the *exponent* of the power.

Thus 3 is the index in the last expression.

**12.** When a quantity is the product of a number of equal factors, each of those factors is called a *root* of that quantity. The order of the root is the same as the number of factors.

Thus, 6 is the second or square root of 36.
And 2 is the fifth root of 32.
And $a$ is the fourth root of $a^4$.

When any root of a quantity is required, it is expressed by writing the sign $\sqrt{\phantom{x}}$ on the left hand of the quantity, and the figure denoting the root required on the left of the sign.

Thus we have $\sqrt[2]{36} = 6$.
$\sqrt[5]{32} = 2$.
$\sqrt[4]{a^4} = a$.

These roots may also be written as follows:—

$\sqrt[2]{36}$ may be written $36^{\frac{1}{2}}$ or $\sqrt{36}$.
$\sqrt[5]{32}$ „ „ $32^{\frac{1}{5}}$.
$\sqrt[4]{a^4}$ „ „ $(a^4)^{\frac{1}{4}}$.

**13.** When the different parts of an expression are separated by the signs $+$ and $-$, each part that is contained between two contiguous signs is called a term.

Thus, $3a - 4b + 4c - 10x^2yz$ contains 4 terms.

**14.** A quantity of *one* term is called a *Monomial.*
A quantity of *two* terms is called a *Binomial.*
A quantity of *three* terms is called a *Trinomial.*
A quantity of more than three terms is called a Multinomial.

Thus, $3xyz$ is a Monomial.
$5a-6xy$ „ Binomial.
$-3x^2+4abc-2$ „ Trinomial.

**15.** The number denoting how many times a quantity is taken is called its coefficient.

Thus, 5 is the coefficient of $5a$.
$-3$ „ „ „ $-3axyz$.

If no coefficient be written, 1 is always understood.

**16.** *Like* quantities are such as differ only in their numerical coefficients.

Thus, $xy$, $4xy$, $-10xy$ are *like* quantities.
But $xy^2$, $x^2y$, $abc$ are *unlike* quantities.

## CHAPTER II.

### NUMERICAL VALUES, ADDITION, ETC.

**17.** The expression $a+b$ represents in general the *sum* of the two quantities $a$ and $b$, and if we suppose $a=4$, $b=3$:

Then, $a+b=4+3$.
Or, $a+b=7$.

∴ 7 is the *numerical value* of $a+b$, when $a$ and $b$ have the values 4 and 3 respectively.

$$\text{Again, } 3a-2b=3\times4-2\times3$$
$$=12-6$$
$$\text{or, } 3a-2b=6.$$

$$\text{And } 3a^2b^2=3\times4^2\times3^2$$
$$=3\times16\times9$$
$$\text{or, } 3a^2b^2=432.$$

If $a$ and $b$ have different values from what we have supposed above, the *numerical* value of the expressions will be found by writing those values in the place of $a$ and $b$ respectively.

Thus, if $a=5$ $b=1$
Then $a+b=6$.

**18.** In order then to find the numerical value of any

Algebraical expression, the number which each letter represents must be known, and the numerical value of the whole expression is found by the following

RULE.

*Write in the expression instead of each letter the number which it represents, and then perform the operations indicated by the signs.*

### Examples I.

(1.) Required the value of $2a+3b-4c$, when $a=2$; $b=4$; $c=3$.

WORKING.

$$\begin{aligned} 2a+3b-4c &= 2\times2+3\times4-4\times3 \\ &= 4 \;+\; 12 \;-\; 12 \\ \therefore\; 2a+3b-4c &= 4. \end{aligned}$$

The following examples should each be worked out in precisely the same form :—

$a=7$; $b=1$: $c=10$.

(2.) $a-10b+2c$ (3.) $a+b+c$ (4.) $3a+3b+3c$
(5.) $10a-10b+c$ (6.) $a+3b-c$ (7.) $-a-b+c$

---

$a=1$; $b=2$; $c=3$; $x=6$; $y=7$; $z=0$.

(8.) $ax+by+cz$ (9.) $ax+by-cz$
(10.) $ab+ac+bc-xy-xz-yz$ (11.) $ax-by$
(12.) $a+bx+cyz$ (13.) $ax+2by-3cz$
(14.) $ayz+10bxz+100cxy$ $a^3y^2z+b^3z^2x+c^3x^2y$.

---

(15.) Let $a=b=5$; $c=2$; $x=y=3$; $z=1$. Find the numerical value of

$$\frac{a^2+b^2+c^2}{x+y+z}.$$

WORKING.

$$\begin{aligned} \frac{a^2+b^2+c^2}{x+y+z} &= \frac{5\times5+5\times5+2\times2}{3+\;3\;+1} \\ &= \frac{25+25+4}{7} \\ &= \frac{54}{7} \\ \therefore\; \frac{a^2+b^2+c^2}{x+y+z} &= 7\tfrac{5}{7} \end{aligned}$$

$a, b, c, x, y, z$ having the same values as in Ex. (15): find the values of the following :—

(16.) $\dfrac{ab+ac+bc}{5}$ (17.) $\dfrac{ac-bc+ab}{x-y+z}$

(18.) $\dfrac{a^2bc-ab^2c+abc^2}{a^2x-b^2y+c^2z}$ (19.) $\dfrac{2a-2b+10c}{x^2-y^2+4z^2}$

(20.) $\dfrac{3abc\ xyz}{a+b+c+x+y+z}$ (21.) $\dfrac{a^2x^2+b^2y^2+c^2z^2}{6abc}$

$a, b, c, x, y, z$ having the same values as in Ex. (8), find the values of the following :

(22.) $\dfrac{a^3x+b^3y}{ax^2+by^2+cz^2}$ (23.) $\dfrac{a^3-b^3}{a-b}$

(24.) $\dfrac{100a^4+b^4-c^4}{b+c}$ (25.) $\dfrac{abxy+bcyz+acxz}{12}$

---

$a=5;\ b=4;\ c=2;\ d=1.$

(26.) $\dfrac{a^3}{b+d} - \dfrac{c^3}{d} + \dfrac{b^2+3c^2}{b+c+d} - 2a+3d$

(27.) $\dfrac{a^2+b^2-d^2}{a+b+d} + \dfrac{abcd}{2b+c} - \dfrac{4a^2-10bc}{2c+d}$

---

*Note.*—$\sqrt{a}$ means the same as $\sqrt[2]{a}$, and is generally called the square root of $a$.

(28.) $\sqrt{5a} + \sqrt{16b} + \sqrt{8c} + 9\sqrt{d}.$

WORKING.

$$\begin{aligned}\sqrt{5a}+\sqrt{16b}-\sqrt{8c}+9\sqrt{d} &= \sqrt{5\times5}+\sqrt{16\times4}-\sqrt{8\times2}+9\sqrt{1}\\ &= \sqrt{25}+\sqrt{64}-\sqrt{16}+9\sqrt{1}\\ &= 5+8-4+9\end{aligned}$$

$$\therefore \sqrt{5a}+\sqrt{16b}-\sqrt{8c}+9\sqrt{d} = 18.$$

(29.) $12\sqrt{bd}-7\sqrt{b}$ (30.) $a^2+4\sqrt{b^2d}$

(31.) $12d^2-4\sqrt{a^2-b^2}$ (32.) $\sqrt[3]{cb}-\sqrt[5]{32d}$

(33.) $\sqrt{a^2d^3}+\sqrt[4]{8b^2c}$

(34.) $\sqrt[3]{25abc}+\sqrt[2]{5abd}-\sqrt[3]{100\ acd}-5^3\sqrt{bcd}$

(35.) $\sqrt{25a^2}+7\sqrt[3]{25a}-\sqrt{81}$

(36.) $\sqrt{\frac{a+b}{a-b}}$; $\sqrt[3]{\frac{b^3-c^3}{29(b-c)}}$; $\sqrt[4]{a^4+4a^2b^2+4b^4}$; $\sqrt[5]{6a+c}$

(37.) $9\sqrt{\frac{a^3+3a^2b+3ab^2+b^3}{a^3-3a^2b+3ab^2-b^3}}$; $3\sqrt[5]{\frac{a^5-1024}{a^4+a^3b+a^2b^2+ab^3+b^4}}$

---

## ADDITION.

**19.** The Addition of Algebraical quantities differs from that of numbers in Arithmetic in one important point, which we shall endeavour to explain.

When several numbers are to be added together in Arithmetic, each of them is regarded as a *positive* or *additive* quantity; that is, as *increasing* the original number to which they are all to be added. Now in Algebra the quantities added together may be either positive or negative, but only *positive* quantities *increase* that to which they are added; whilst the *negative* quantities diminish it.

This principle may be understood from the following instances:—

(*a.*) If a man's property be 500*l.*, and his debts 200*l.*, we may represent algebraically

His property by $+500$ and
His debts by $-200$

and the *adding* of these together algebraically would give $+300$, meaning that the whole *sum* or net result is *property* to the amount of 300*l.*

(*b.*) If a trader gain 40*l.* by one transaction and lose 50*l.* by another, we must represent their sum or combined effect upon his affairs by

$+40-50$
or $-10$*l.*

This expression denoting a *loss* of 10*l.*

(*c.*) If a man be weighed by means of a pair of scales and weights, and finds when the scales are even that there are 2 cwts. in the scale opposite to him, and 37 lbs. in his own, his weight in lbs. will be represented by

$+224-37$
or $+187$ lbs.

From these examples it will be seen that quantities which are the exact opposites of each other (as *property* and *debt*, *gain* and *loss*, time *past* and time *future*, &c.) must be represented by the opposite signs $+$ and $-$. Also that the quantities to be added together may be either positive or negative.

**20.** DEFINITION.—*To add together a number of like Algebraical Quantities is to express in a single term the result of all the separate additions and subtractions. This result is called the sum.*

Ex. 1. Add together $x$, $3x$, and $5x$.

Here all the quantities have positive signs, therefore they are all to be added. Now it is clear that to add $1x$, $3x$, and $5x$ successively is the same thing as to add $9x$ at once; or the sum of $x$, $3x$, and $5x$ is $9x$.

Ex. 2. Add together $-x$, $-3x$, $-5x$.

Here all the quantities have negative signs, therefore each of them must be subtracted. Now it is clear that to subtract $1x$, $3x$, and $5x$ successively is the same thing as to subtract $9x$ at once; or the sum of $-x$, $-3x$, and $-5x$ is $-9x$.

Ex. 3. Add together the following:

$$x-4x+11x-10x-x-7x+5x.$$

Here the quantities to be added are $x$, $11x$, $5x$, or $17x$; and those to be subtracted are $4x$, $10x$, $x$, $7x$, or $22x$; or the effect of all the seven quantities is to add $17x$, and to subtract $22x$, and since the number of $x$'s subtracted is greater by 5 than the number of $x$'s added, the final result is that $5x$ must be subtracted. The sum of the whole therefore is $-5x$.

**21.** From the above explanations, we may deduce the following Rule for the Addition of Algebraical Quantities.

### RULE.

(a.) *Write the like quantities in vertical columns.*

(b.) *In the left-hand column of quantities first add together all the positive coefficients, then all the negative coefficients: subtract the less of these two numbers from the greater. Write down the number that remains in the same vertical line, and after it write the letter or letters that belong to each quantity in the column, and before it place the sign that belongs to the greater of the two numbers.*

(c.) *Apply rule* (b) *to every other column.*

## Examples II.

(1.) Add together $7a+2b-3c$; $12a-9b+2d$; $8b-11d$; $13c$; $a-b+c-d$; $-3c-3d$.

WORKING.

| By Rule (a.) | | | | |
|---|---|---|---|---|
| | $7a$ | $+2b$ | $-3c$ | * |
| | $12a$ | $-9b$ | * | $+2d$ |
| | * | $+8b$ | * | $-11d$ |
| | * | * | $+13c$ | * |
| | $a$ | $-b$ | $+c$ | $-d$ |
| | * | * | $-3c$ | $-3d$ |
| | $20a$ | * | $+8c$ | $-13d$ |

| By Rule (b.) | 1st column. | 2nd column. | | 3rd column. | | 4th column. | |
|---|---|---|---|---|---|---|---|
| | 7 | 2 | −9 | 13 | −3 | +2 | −11 |
| | 12 | 8 | −1 | 1 | −3 | | −1 |
| | 1 | — | — | — | — | | −3 |
| | — | +10 | −10 | +14 | −6 | | — |
| | +20−0=20 | 10 | | 6 | | | −15 |
| | | — | | — | | | 2 |
| | | 0 | | +8 | | | — |
| | | | | | | | −13 |

(2.) Add together $x+y$; $-3x+7y$; $4x-7y$.

(3.) Add together $7a-3b+10$; $-8a+4b-11$; $12a-14$; $-a-b-1$.

(4.) Find the sum of $3a^2x^3-4ax^2$; $-3x$; $5a^2x^3-2ax^2-x$; $4a^2x^3-ax^2-5x$; $-a^2x^3+4x$.

ARRANGEMENT.

$$\begin{array}{rrr}
3a^2x^3 & -\ 4ax^2 & -\ 3x \\
5a^2x^3 & -\ 2ax^2 & -\ x \\
4a^2x^3 & -\ ax^2 & -\ 5x \\
-a^2x^3 & & +\ 4x.
\end{array}$$

*Note.*—Here it must be noticed that the quantities in the first and second columns are *unlike*, since the indices of the letters are different, although the letters themselves are the same in both.

(5.) Find the sum of $4x^3-9ax^2+11a^2x+3a^3$; $2x^3-11ax^2+9a^2x+a^3$; $-13ax^2+7a^2x-a^3$; $-15ax^2+5a^2x+3a^3$.

(6.) $2+m-3n$ and $m+n-3$.

(7.) $3x-2z$; $4z-5x$; $2x-2z$.

(8.) $3ax^5-a^3x^3+7x^6$; $-x^6+3a^2x^4-9a^4x^2$; $3a^5x+3a^3x^3-7a^4x^2$; $ax^5-9a^2x^4$; $4a^6+a^5x+20a^4x^2+2a^3x^3+10a^2x^4-2x^6$.

(9.) $xy+4z$; $3xy-z$; $-4xy-3z$.

(10.) $3x^5y-4x^3y^3+xy^5$; $x^5y+3x^3y^3+3xy^5$; $-3x^5y+x^3y^3-6xy^5$.

(11.) $1-x$; $2x-2$; $x+4$; $9-x$.

(12.) $3a^2-b^2$; $2ab-2a^2$; $2b^2-ab$.

Find the sums of the following :—

(13.) $a^2b^2c^2-4a^6+6a^4c^2-c^6$; $6a^2c^4-b^4c^2+a^2b^4$; $b^6-a^2c^4+7a^6-9a^2b^2c^2+4b^4c^2$; $-3a^2b^4-9a^4c^2+a^2b^2c^2-3a^2c^4+11b^2c^4$.

(14.)
$-a-2b-3c-4d$
$-2a-3b-4c-d$
$-3a-4b-c-2d$
$-4a-b-2c-3d$

(15.)
$1-4a-5b^2$
$a-4+7b^2$
$-3+a-12b^2$
$6+2a+10b^2$

(16.)
$3a^4-7a^3x+10a^2x^2-17ax^3+9x^4$
$-4a^4+17x^4-19ax^3+17a^2x^2-a^3x$
$31x^4-19a^4+7a^3x-18ax^3+a^2x^2$
$-57x^4+54ax^3-28a^2x^2+a^3x \quad 20a^4$.

(17.)
$a^3-abc+2b^3$
$b^3-abc+2c^3$
$c^3-bca+2a^3$.

(18.)
$m^3+m^2n-mn^2+3n^3$; $-7m^3+10n^3$;
$3n^2m-3nm^2+5n^3$; $17m^3-24mn^2$:

(19.)
$-7p^2qr+10pq^2r-pqr^2$
$14q^2rp-18qp^2r+11pqr^2$
$-qp^2r+7rp^2q-11p^2rq$
$-7q^2pr-r^2qp+p^2qr$
$10r^2pq-q^2rp-17pq^2r$.

Add together the following :—

(20.)
$\frac{1}{3}x^2-\frac{1}{2}xy+2y^2$
$\frac{2}{5}x^2+\frac{5}{2}xy-3y^2$
$\frac{5}{6}x^2-\frac{1}{4}xy+10y^2$.

*Working.*—The coefficients are partly fractional, therefore the rules for the addition and subtraction of fractions must be employed.

Hence we have for 1st column - $\frac{1}{3}+\frac{2}{5}+\frac{5}{6}=\frac{10+12+25}{30}=\frac{47}{30}$

" " 2nd column - $+\frac{5}{2}$ the positive

" " and $\frac{1}{2}+\frac{1}{4}=\frac{2+1}{4}=\frac{3}{4}$ the negative

$\therefore \quad \frac{5}{2}-\frac{3}{4}=\frac{10-3}{4}=\frac{7}{4}$.

And the whole sum is $\frac{47}{30}x^2+\frac{7}{4}xy+9y^2$.

(21.)
$a^2-9b^2+\frac{1}{9}c^2$
$-\frac{1}{3}a^2+\frac{4}{3}b^2-c^2$
$-\frac{2}{3}a^{\frac{2}{9}}-\frac{5}{6}b^2+\frac{2}{9}c^2$
$4a^2-\frac{1}{6}b^2+\frac{2}{3}c^2$

(22.)
$5am-6bn+7cp-1$
$\frac{1}{5}am-\frac{1}{6}bn-\frac{1}{7}cp+4$
$\frac{3}{10}am-\frac{5}{12}bn+\frac{2}{7}cp-3$
$-\frac{1}{2}am-\frac{5}{12}bn-\frac{3}{7}cp+9$

(23.)
$$\begin{array}{r}
ayz - bxz + cxy \\
-7ayz + 4bxz - 3cxy \\
8ayz - 9bxz + 4cxy \\
11ayz + 12bxz - 3cxy \\
\hline
\end{array}$$

(24.)
$$\begin{array}{r}
8m - 6n + 3p + 4q \\
\frac{m}{6} - \frac{3}{5}q + n - \frac{5p}{8} \\
\frac{q}{4} - \frac{n}{5} + p - m \\
-\frac{p}{2} + \frac{m}{3} - \frac{p}{4} + q \\
2m - \frac{2}{3}n - \frac{2}{5}q + \frac{1}{2}p \\
\hline
\end{array}$$

(25.)
$$\begin{array}{r}
\frac{2}{3}a^3 + \frac{2}{3}b^3 - \frac{1}{3}c^3 - abc \\
\frac{2}{3}b^3 + \frac{2}{3}c^3 - \frac{1}{3}a^3 - abc \\
\frac{2}{3}c^3 + \frac{2}{3}a^3 - \frac{1}{3}b^3 - bac \\
\hline
\end{array}$$

# CHAPTER III.

## SUBTRACTION.

**22.** The principle upon which Subtraction is performed in Algebra is this :—Subtracting a positive quantity is the same as adding a negative quantity; and Subtracting a negative quantity is the same as adding a positive quantity.

This may be understood from the illustration given in page 7: for taking a man's property to be a positive quantity, and his debts to be negative quantities, it is clear that to take 10*l.* of property away is the same in effect as to add 10*l.* to his debts; and to take away a debt of 10*l.* is the same in effect as to add 10*l.* to his property.

$$\left.\begin{array}{ll}\text{And generally to subtract} & +a \\ \text{is the same as to add} & -a\end{array}\right\}$$
$$\left.\begin{array}{ll}\text{Also to subtract} & -a \\ \text{is the same as to add} & +a\end{array}\right\}$$

Hence is derived the Rule for Subtraction of Algebraical quantities.

### RULE.

*Change the signs of all the quantities to be subtracted, and then proceed as in Addition.*

Ex. (1.) Subtract $a+b-c$ from $2a+b+3c$.

*Working.*—The quantities to be subtracted are $+a$, $+b$, and $-c$; changing their signs they become $-a$, $-b$, and $+c$; and writing them as now changed under the other line, we have to add together the two lines :—

$$\begin{array}{r}
2a + b + 3c \\
-a - b + c \\
\hline
a \qquad + 4c
\end{array} = \text{Remainder.}$$

Ex. (2.) From $6x$ take $3x+3y-3z-3$.

Changing the signs as before, we have—

$$\begin{array}{l} 6x \\ -3x-3y+3z+3 \\ \hline 3x-3y+3z+3 = \text{Remainder.} \\ \hline \end{array}$$

*Note.*—At first the beginner should be very careful to write down the lower line with changed signs as in the above examples. After he is perfectly sure of the rule, the trouble of *writing* the lower line with changed signs may be spared: it will be sufficient for him to *conceive* the signs to be changed, and then to proceed as in addition.

### Examples III.

(3.) From $2x+3y$ take $x-y$ (4.) From $-6x+3y$ take $-x+3y$

(5.) From $x+y$ take $-x-y$ (6.) From $-3x-4y$ take $3x-4y$

(7.) From $2x+y-1$ take $y+1$ (8.) From $x^2+2$ take $3x-2$

(9.) From $x^3+1$ take $x^2+1$ (10.) From 4 take $x+8$

(11.) From $3x$ take $x^2+3x-3$ (12.) From 4 take $-2$.

(13.) Subtract $a-b-c$ from $a+b+c$

(14.) Subtract $3a^3-5a^2x-3ax^2-3x^3$ from $5a^3-4a^2x+2ax^2-3x^3$

(15.) Subtract $-2a^3+4a^4-6a^5$ from $a^3-4a^4+6a^5$

(16.) Subtract $a^3-4a^2x-ax^2-x^3$ from $3a^3-3a^2x+ax^2-2x^3$

(17.) Subtract $x^5y+3x^3y^3+3xy^5$ from $3x^5y-4x^3y^3+xy^5$

Find the excess of the upper line over the lower line in the following examples:—

(18.) $\begin{array}{r} 5x^4z^2-a^6 \\ x^4z^2-a^6 \\ \hline \end{array}$ (19.) $\begin{array}{r} x^3+3x^2y+3xy^2+y^3 \\ -x^3+3x^2y+3xy^2-y^3 \\ \hline \end{array}$

(20.) $\begin{array}{r} 4a^3+ab^2 \\ a^2b-b^3 \\ \hline \end{array}$ (21.) $\begin{array}{r} 3a^3-7a^4+15a^5 \\ 2a^3-3a^4+4a^5 \\ \hline \end{array}$

(22.) Find the difference between $a^2+2a+3$ and $a+3$

(23.) " " " $8xy-4yz$ and $2yz-4xz$

(24.) " " " $a^2+a+1$ and $a^3-1$

(25.) " " " $x^4+x^3y+x^2y^2+xy^3+y^4$
and $x^4-4x^3y+6x^2y^2+4xy^3+y^4$

(26.) Find the difference between the sum of $a+b$ and $a-b$; and the sum of $2a+3b$ and $5a-6b$

(27.) Subtract the difference of $x$ and $y$ from $4xy+x+y$

(28.) Subtract 5 times the square of $y$ from $4y^2+1$

(29.) Subtract $4x^2-y^3-4$ from $x^2-y^3$

(30.) Subtract $x^4+4x^3y+6x^2y^2+4xy^3+y^4$
from $x^4-4x^3y \qquad\qquad +2y^4$

(31.) From $a+\frac{2}{3}b-1$
take $\frac{1}{2}a-\frac{1}{4}b+\frac{1}{3}$

*Working.*

$$\begin{array}{r} a+\frac{2}{3}b-1 \\ -\frac{1}{2}a+\frac{1}{4}b-\frac{1}{3} \\ \hline \\ \hline \end{array}$$

Having changed the signs of the quantities in the lower line, we find by addition that the

Coefficient of $a$ in the result is $1-\frac{1}{2}=\frac{1}{2}$
" $b$ " $\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$
and the last term is $-1-\frac{1}{3}=-\frac{4}{3}$

Or the whole remainder $=\frac{1}{2}a+\frac{11}{12}b-\frac{4}{3}$

(32.) From $a+b+3$
take $a-\dfrac{b}{3}-\frac{1}{2}$

(33.) From $\frac{1}{2}x+\frac{1}{2}y$
take $a+y$

(34.) From $\dfrac{x}{2}+1$
take $3x-\frac{1}{2}$

(35.) From $x^4+\frac{1}{4}x^3+\frac{1}{6}x^2+\frac{1}{4}x+1$
take $x^4-\frac{1}{4}x^3+\frac{1}{6}x^2-\frac{1}{4}x+1$

(36.) From $a$
take $\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}c$

(37.) From $abc+ab+a-\frac{1}{4}$
take $\frac{3}{4}abc-ab+a+\frac{1}{4}$

(38.) From $\frac{1}{3}a^3+1$
take $\frac{1}{3}a^2+a$

(39.) Find the difference between $\frac{1}{2}p^2+\frac{1}{3}pq-\frac{1}{4}q^2$ and $-\frac{1}{2}p^2-\frac{1}{3}qp+\frac{1}{4}q^2$

(40.) Between $4\frac{2}{7}a^2-\frac{1}{3}ab+3\frac{1}{4}b^2$ and $1\frac{3}{7}a^2+\frac{2}{3}ab-\frac{1}{4}b^2$

## BRACKETS.

**23.** When any operation is to be performed upon a quantity consisting of several terms, it is often found convenient to enclose the quantity within brackets; and the quantity thus enclosed is now to be regarded as a whole, and the operation indicated is to be performed upon it as a whole.

Thus, to express the double of the quantity .. .. .. $a+b$
it is first enclosed in brackets .. .. .. .. $(a+b)$

And if the number 2 be written before it thus .. .. $2(a+b)$
we have expressed the double of the whole quantity .. $(a+b)$

Also, since the double of the total quantity $(a+b)$ is the sum of the doubles of each separate part, we have

$$2(a+b)=2a+2b$$

Similarly $3(a+b+c)=3a+3b+3c$

Next, let us consider the bracketed quantity $5(a-b)$. Now the value of $5(a-b)$ may be found by writing down $a-b$ five times, and adding all these quantities together: thus,

$$\begin{array}{c} a-b \\ a-b \\ a-b \\ a-b \\ a-b \\ \hline 5a-5b \\ \hline \end{array}$$

$$\therefore\ 5(a-b)=5a-5b$$

In the same way the value of $x(a-3b+2)$ will be found to be

$$ax-3bx+2x$$

for the line $a-3b+2$ must be repeated $x$ times, and all these lines must be added together.

---

**24.** From these examples we may derive the following Rule for the multiplication of a bracketed quantity.

RULE.

*When a bracketed expression is multiplied by any positive quantity, the result is obtained by multiplying the coefficient of each of the terms by that quantity and removing the bracket.*

### Examples IV.

Remove the brackets from the following expressions without altering their values:—

(1.) $4(a-2b-3c+x)$

*Working.* $4(a-2b-3c+x)=4a-4\times 2b-4\times 3c+4x$

$\qquad =4a-8b-12c+4x$

(2.) $3(1-4x)$
(3.) $5(x^2-y^2)$
(4.) $7(x^2+y^2+z^2)$
(5.) $(3x-4y+4z)$
(6.) $a(b+c)$
(7.) $x(y+z-a)$
(8.) $2a(1+x+x^2)$
(9.) $3x(x^2+x+1)$
(10.) $4(ab+ac+bc)$
(11.) $ab(a+b-1)$
(12.) $12abc(a+b+c)$
(13.) $xyz(x^2-y^2-z^2)$
(14.) $a(a+b+c)$
(15.) $x^2y^2(x^2-y^2)$
(16.) $10x(10x-5y)$
(17.) $11(xyz-abc)$
(18.) $3b^2(3ab+4b^2)$
(19.) $3a^2b^2(ab-ac-3bc)$

**25.** We will now consider the meaning of a bracketed expression when preceded by a negative sign, as

$$-2(a+b)$$

Now the minus sign prefixed to this quantity denotes that *the whole of it is to* be subtracted.

Now, as we have seen above, $2(a+b)=2a+2b$

and both the parts $2a$ and $2b$ must be subtracted; and we have

$-2a-2b$ for the whole value;

or $-2(a+b)=-2a-2b$.

Again, let it be required to remove the brackets from the expression

$$-4(a-3b-2c)$$

Now $4(a-3b-2c)=4a-12b-8c$

and to express the subtraction of each of these terms, we must apply the Rule of Subtraction, and change the sign of each of them:

Thus we have $-4(a-3b-3c) = -4a+12b+8c$.

**26.** Hence we may derive the following Rule for removing the brackets from a quantity preceded by a negative sign.

RULE.

*When a bracketed expression is preceded by a negative quantity, the result is obtained by multiplying each of the coefficients of the included terms by that quantity, removing the brackets, and changing the signs of every term.*

### Examples V.

Remove the brackets from the following expressions:—

(1.) $2(a+3b)-6(a-b)+7a-(a+b)$

WORKING.

$$\begin{aligned}2(a+3b)-6(a-b)+7a-(a+b) &= 2a+6b-6a+6b+7a-a-b\\ &= 2a+11b\end{aligned}$$

(2.) $7(a-4x)$ (3.) $(a+b)-(a-b)$

(4.) $2(a^2+ab+b^2)-2(a^2-ab+b^2)$ (5.) $(a-b)x^2$

(6.) $3b(a^2-2a+1)$ (7.) $4xy(x^2-y^2)$

(8.) $-(a+b)ab+(a-b)a^2$ (9.) $(a-c)x^2-(b-d)y^2$

(10.) $(a+b+c)xy-(ax-bx+cx)y-(1-2by)x$

(11.) $6(a^4+b^4)-a^2(6a^2-7b^2)+a(b^2-a^3)$

(12.) $2(x^2+3y^2)-3(x^2-2y^2)+x^2(1-y^2)$

(13.) $3xy(ab-b^2)-3ab(xy-ab)$

(14.) $4(x+2)-2(2x+4)$ (15.) $a(x^2+xy+y^2)-a(x^2-xy+y^2)$

(16.) $3ab(x^2-2xy+y^2)-3xy(a^2-2ab+b^2)$

(17.) $(a+b+c)-(a+b-c)-(b+c-a)-(c+a-b)$

(18.) $xyz(x+y+z)-yz(xz+xy)-xy(yz+xz)-xz(xy+zy)$

**27.** If we wish to express that the sum of $a$ and $b$ is to be multiplied by the sum of $b$ and $c$, each factor must be formed by bracketing.

Then $(a+b)$ is the first factor,
and $(b+c)$ is the second factor;
And their product is $(a+b)\times(b+c)$;
Or more simply $(a+b)\,.\,(b+c)$.

Similarly the second power or square of the difference of $a$ and $b$ is written

$$(a-b)^2$$

And to express the subtraction of the square of $(a+b)$ from three times the difference of $a$ and $b$, we must write

$$3(a-b)-(a+b)^2$$

Let it be required to determine the numerical value of the last expression, when $a=2$ $b=1$.

$$\begin{aligned} 3(a-b)-(a+b)^2 &= 3(2-1) - (2+1)^2 \\ &= 3\times1 \quad - 3^2 \\ &= \;\;3 \quad\;\; - 9 \\ &= -6 \end{aligned}$$

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## Examples on the Numerical Values of Bracketed Expressions, VI.

Let $a=7$; $b=3$; $c=4$; $x=y=2$; $z=10$

Find the value of the following :—

(1.) $4(a+b)+3(a-b)-(3a+2b)$ (2.) $10(a+b-10)+4(x+y)+abe$
(3.) $(a-b)(x+z)-(c-b)(y+z)$ (4.) $(a+b)x+(a+c)y+(b+c)z$
(5.) $3(a+b)^2-2(a-b)^2$
(6.) $(a+b)(b+c)(a+c)-12abc+6xyz$
(7.) $2(ax-by)+3(cz-ax)-(cz-by)$
(8.) $c(a-b)^2+b(a-c)^2+a(c-b)^2$ (9.) $(a+b+c)^3-(a^3+b^3+c^3)$
(10.) $(a+b)^2+(a+c)^2+(b+c)^2-(a+b+c)^2$
(11.) $\sqrt{21a+26b}$ (12.) $\sqrt{4c-6x}$
(13.) $xy(\sqrt{7a}+\sqrt{4c}+\sqrt{3b})-ab(\sqrt{xy}-\sqrt{10z}+\sqrt{cxy})$
(14). $2\sqrt{a+z-1}+3\sqrt{a-b}-\sqrt{2b-a}$
(15.) $\sqrt[3]{4c-4y}-4\sqrt[3]{2c-a}+\sqrt[3]{10z+4a-b}$
(16.) $ab\sqrt{c-3}-bc\sqrt{a+2}+ac\sqrt{27b}$

# CHAPTER IV.

## ON MULTIPLICATION.

**28.** *To multiply an Algebraical Quantity by any number is to repeat that quantity as many times as the number expresses; then to add the result, if the multiplier be positive, but to subtract it, if the multiplier be negative.*

We shall now consider four different cases of Multiplication.

1°. To multiply $+a$ by $+4$ is to repeat $+a$ 4 times, and add the result.

Now $+a$ repeated 4 times gives $+4a$ as appears from the accompanying addition:

$$\begin{array}{r} a \\ a \\ a \\ a \\ \hline 4a \\ \hline \end{array}$$

And to add the result, $4a$ must have a positive sign.

$$\therefore\ +a \times +4 = +4a.$$

2°. To multiply $-a$ by $+4$ is to repeat $-a$ 4 times, and add the result.

Now $-a$ repeated 4 times gives $-4a$:

$$\begin{array}{r} -a \\ -a \\ -a \\ -a \\ \hline -4a \\ \hline \end{array}$$

And to add the result, $-4a$ must keep the negative sign.

$$\text{Or } -a \times +4 = -4a.$$

3°. To multiply $+a$ by $-4$ is to repeat $+a$ 4 times, and subtract the result.

$$\begin{array}{r} a \\ a \\ a \\ a \\ \hline +4a \\ \hline \end{array}$$

The result is $+4a$, and to subtract this we must change its sign; it then becomes $-4a$.

$$\text{Or } +a \times -4 = -4a.$$

4°. To multiply $-a$ by $-4$ is to repeat $-a$ 4 times, and to subtract the result.

$$\begin{array}{r} -a \\ -a \\ -a \\ -a \\ \hline -4a \\ \hline \end{array}$$

The result is $-4a$, and to subtract this we must change its sign; it then becomes $+4a$.

$$\text{Or } -a \times -4 = +4a.$$

**29.** If instead of 4 times, we suppose $a$ to be repeated $b$ times, the 4 cases may be gone through in the same order as before, and the proofs will then be true for the multiplication of two factors $a$ and $b$; $a$ and $b$ being any numbers whatever.

We shall then have the following general results:—

$$\begin{array}{ll} 1^\circ. & +a \times +b = +ab \\ 2^\circ. & -a \times +b = -ab \\ 3^\circ. & +a \times -b = -ab \\ 4^\circ. & -a \times -b = +ab. \end{array}$$

These rules for the sign of the product of two quantities may be expressed in the following words:—

*The product of two quantities of the same signs is positive, and the product of two quantities of different signs is negative.*

Ex. 1. Multiply together $3a$ and $4bc$.

By the definition of Multiplication, $3a$ is to be repeated $4bc$ times that is, $12a$ is to be repeated $bc$ times; that is, $12ab$ is to be repeated $c$ times, and the result of this last operation is evidently $12abc$.

Ex. 2. Multiply together $-5a^3$ and $6a^2b$.

By the same method as in the last example it may be shown that the product is $-30a^3.a^2b$

$$\begin{aligned} \text{But} \qquad a^3.a^2 &= a.a.a \times a.a \text{ by Def. (11.)} \\ &= a^5 \\ \therefore \; -5a^3 \times 6a^2b &= -30a^5b \end{aligned}$$

**30.** From the working of the last two examples, we may deduce the following Rule for the Multiplication of two simple Algebraical Quantities.

### ($a$.) RULE FOR THE SIGN OF THE PRODUCT.

*When the factors have each the same sign, the sign of the product will be positive; but when the factors have different signs, the sign of the product will be negative.*

### ($b$.) RULE FOR THE COEFFICIENT OF THE PRODUCT.

*The coefficient of the product is found by multiplying together the coefficients of the two factors.*

### (c.) RULE FOR THE INDEX.*

*The Index of each letter of the product is found by adding together the indices of that letter in the two factors.*

---

## Examples VII.

(1.) Find the product of $3ac$ and $12a^2b$, of $-5ac$ and $10ac^2$, of $-abc$ and $abc$.

(2.) Multiply together the following:—

$3x^2y$ and $-5$ | $-xy^2z$ and $-x^2z^2$
$-7xy^2$ and $-7x^2y$ | $4xy$ and $-4x^2y^2$
$4x^2z$ and $4xz^2y^3$ | $-5a^2y^2$ and $5b^2x^2$
$-a^8b^8c^6$ and $-a^6b^8c^9$ | $3x^3y^2z$ and $-2y^2z^3$

*Note.*—If any of the coefficients be fractional, they must be multiplied together by the rule for the multiplication of fractions.

Thus, $\frac{1}{2}xy \times \frac{1}{3}x^2 = \frac{1}{6}x^3y$.

(3.) Find the products of the following:—

$10xy \times \frac{1}{5}y^2$ | $-1\frac{3}{7} \times 2\frac{1}{3}b^2$ | $-20a^2b^2 \times \frac{3}{5}c^2d^2$
$-\frac{3}{5}x^2 \times \frac{5}{6}y^2$ | $2\frac{1}{2} \times 5b^2c^2$ | $\frac{9}{4}pr \times -\frac{4}{9}qr$
$-2\frac{2}{3}ab \times 1\frac{1}{4}b^3$ | $\frac{3}{2}abc \times \frac{3}{2}ab$ | $\frac{9}{4} \times 2\frac{1}{4}$
$-\frac{1}{7}a^2c^2 \times 14a^3c^3$ | $9a \times \frac{1}{3}a^2$ | $\frac{1}{3} \times 3x^2y^2$.
$-\frac{1}{2}ad \times -\frac{1}{4}bc$ | $-\frac{1}{4}a^3 \times -4b^3$

(4.) Multiply $a-b$ by $3ab$

WORKING.

$$\begin{aligned}(a-b)\,3ab &= 3ab \times a - 3ab \times b \\ &= 3a^2b - 3ab^2\end{aligned}$$

Examples of this form must be worked by multiplying each term of the compound factor in succession by the simple factor.

(5.) Multiply

$2ab+b^2$ by $ab$
$a-b+c$ by $-3abc$
$3a+3b+3c$ by $2$
$xy+xz+yz$ by $-xyz$
$a^2+ab+b^2$ by $3ab$
$x^3-3x^2y+3xy^2-y^3$ by $4x^3$
$4x^2-4y^2$ by $\frac{1}{4}xy$

Multiply

$\frac{x^2}{2} - \frac{xy}{3} + \frac{y^2}{4}$ by $-12$
$10a^2-7ab-b^2$ by $-\frac{1}{3}a^2b^2c^2$
$x^3-\frac{3}{2}x^2y+\frac{3}{4}xy^2-\frac{1}{8}y^3$ by $8$
$\frac{1}{4}p^2-\frac{3}{2}pq+4$ by $-\frac{1}{8}q^2$
$x^2yz-xy^2z+xyz^2$ by $\frac{3}{5}x^2y^3z^4$
$-1+x^2+x^4$ by $-2x^2$
$-p^2q^2-p^2r^2-q^2r^2$ by $-\frac{1}{4}p^2qr^3$

(6.) Find the product of $3a+b$ and $2a-b$

---

* See proof of (c) at p. 22.

WORKING.

$$\begin{aligned}(3a+b)\times(2a-b) &= (3a+b)\times 2a-(3a+b)\times b\\ &= 3a.2a+b.2a-3a.b-b.b\\ &= 6a^2+2ab-3ab-b^2\\ &= 6a^2-ab-b^2\end{aligned}$$

In examples like this, in which the two factors are compound quantities, each term of the one factor must be multiplied into each term of the other; and the work is generally arranged in this manner :—

$$\begin{array}{lll} 3a & +\ b & \\ 2a & -\ b & \\ \hline 6a^2 & +\ 2ab & \\ & -\ 3ab & -\ b^2 \\ \hline 6a^2 & -\ ab & -\ b^2 \\ \hline \end{array}$$

(7.) Multiply together $a^2+ab+b^2$ and $3b^2-6ab+3a^2$ :—

$$\begin{array}{lllll} a^2 & +\ ab & +\ b^2 & & \\ 3a^2 & -\ 6ab & +\ 3b^2 & & \\ \hline 3a^4 & +\ 3a^3b & +\ 3a^2b^2 & & \\ & +\ 6a^3b & -\ 6a^2b^2 & -\ 6ab^3 & \\ & & 3a^2b^2 & +\ 3ab^3 & +\ 3b^4 \\ \hline 3a^4 & -\ 3a^3b & * & -\ 3ab^3 & +\ 3b^4 \\ \hline \end{array}$$

*Note.*—Observe, in the work the two quantities are arranged according to the powers of $a$. That is the highest powers of $a$ are placed first, then the next powers of $a$; the answer is found to be arranged in the same order, and this rule of placing the factors should always be carefully attended to.

(8.) Find the product of the following factors.

$1-2x+3x^2-4x^3$ and $1+x$
$1+a+a^2$ and $a-2a^2+3a^3$
$2a+b$ and $2a-b$
$5x-4y$ and $3x-2y$
$x^2+2bx+b^2$ and $x^2-2bx+b^2$
$x^3+x^2y+xy^2+y^3$ and $x-y$
$a^4-a^3x+a^2x^2-ax^3-x^4$ and $a+x$
$x^3y^3+x^2y^2+xy+1$ and $xy-1$
$a^3-6a^2+12a-8$ and $a-2$
$a^4b^4+2a^3b^3+4a^2b^2+8ab+16$ and $ab-2$
$a^2b^2-ab+1$ and $1+ab+a^2b^2$
$a^3-3a^2p+3ap^2-p^3$ and $p^2-2ap+a^2$

(9.) Find the square of the following quantities :—

$(3x-1)$; $(a-b)$; $7x+5y$; $a^2+1$; $a^2-b^2$; $1-x^3$; $ab+cd$; $4ax-8by$; $1-x+x^2$; $x-x^2+1$; $ay-by^3+1$; $c^3-1+c$.

(10.) Cube the following quantities :—

$x+1$; $a+b$; $1-x$; $2x+a$; $3x^2+y^2$; $7a-1$; $2a+3b$; $x^2y-xy^2$; $9-a^2$; $x^2-x+1$.

(11.) Find the continued product of $a+1$, $a-1$, $a^2+1$, and $ab$

*Working.*

$$\begin{array}{lll}
a & +1 & \\
a & -1 & \\
\hline
a^2+a & & \\
\phantom{a^2}-a & -1 & \\
\hline
a^2 & -1 & \\
a^2 & +1 & \\
\hline
a^4 & +a^2 & \\
 & \phantom{+}a^2 & -1 \\
\hline
a^4 & & -1 \\
ab & & \\
\hline
a^5b & & -ab \quad \textit{Ans.} \\
\hline
\end{array}$$

(12.) Find the continued product of the following :—

| | |
|---|---|
| $3a$, $4b$, $7a^2$ and $-ab$ | $x^2+xy+y^2$, $x^2xy+y^2$, $(x+y)^2$ |
| $(a+b)$, $a-b$, $a^2+b^2$ | $a^2-a+1$, $a^2+a+1$, $(a-1)^2$ |
| $x+1$, $x-1$, $x-2$, $x-2$ | $ab-1$, $ab+1$, $a^2b^2+1$, $a+b$ |
| $4x$, $3y$, $2(a-b)$, $3(a+b)$ | $a^2+b^2$, $a^2-b^2$, $(a+b)^2$, $(a-b)^2$ |
| $x-2$, $x-1$, $x$, $x+1$, $x+2$ | $a+2$, $a+3$, $a+4$, $a-a$ |
| $a+b+c$, $a+b-c$, $b+c-a$, $a+c-b$ | $5a-a$, $5a-3a$, $5a-5a$, $5a-7a$ |

(13.) Multiply $\frac{a}{2}-\frac{b}{3}$ by $2a+3b$

*Working.*

$$\begin{array}{rll}
 & \frac{a}{2}-\frac{b}{3} & \\
 & 2a+3b & \\
\hline
 & \frac{2}{2}a^2-\frac{ab}{2\times 3} & \\
 & \phantom{\frac{2}{2}a^2-{}}\frac{3}{2}ab-\frac{3}{3}b^2 & \\
\hline
 & a^2-\frac{ab}{6}+\frac{3}{2}ab-b^2 & \\
= & a^2+\frac{4}{3}ab \qquad -b^2 & \\
\hline
\end{array}$$

*Note.*—In working examples with fractional coefficients, the same rules must be observed as in the case of integral coefficients, the fractions being added or multiplied according to the methods of vulgar fractions.

(14.) Multiply

$4a-b$ into $a-\frac{b}{4}$

$\frac{1}{2}a^2+\frac{1}{2}b^2$ into $4ab$

$a^2-\frac{ab}{3}+\frac{b^2}{9}$ into 27

Multiply

$x-\frac{2}{3}$ into $\frac{2}{3}x-1$

$a^2+\frac{1}{2}a+\frac{1}{4}$ into $\frac{1}{9}+\frac{1}{3}a+a^2$

$x-\frac{3}{4}y$ into $x-\frac{1}{3}y$

(15.) Find the squares of the following :—

$$\tfrac{1}{2}a-1;\ \frac{b}{3}+6;\ \frac{a}{4}-\frac{b}{3};\ pq-\tfrac{1}{2};\ a^2+\tfrac{1}{2}a+\tfrac{1}{4};\ \tfrac{2}{3}a-\tfrac{3}{2}b;\ 3a^2-\frac{b^2}{3};$$

$$x-\frac{x}{4}+\frac{x}{3};\ -\frac{y}{2}-\frac{x}{2};\ \frac{a}{2}+\frac{b}{3}+\frac{c}{4};\ -3a^2+ab-\frac{b^2}{3};\ \frac{5}{6}x^2-xy+\frac{6}{\ }y^2$$

(16.) Cube the following quantities :—

$$a-\tfrac{1}{2};\ 3a^2-\tfrac{1}{4};\ \tfrac{1}{2}a+\tfrac{1}{2}b;\ \frac{x}{3}-\frac{y}{2};\ \frac{x}{1}-\frac{y}{2};\ \frac{4x}{2}-\frac{14y}{7}$$

### PROOF OF RULE (c), p. 19.

**31.** *When two powers of the same letter are multiplied together, the Index of the product is found by adding together the Indices of the two factors, that is :—*

$$a^m \times a^n = a^{m+n}$$

For $a^m = a.a.a.a\ldots\ldots$ to $m$ factors by def.

And $a^n = a.a.a.a\ldots\ldots$ to $n$ factors by def.

Then $a^m \times a^n = (a.a.a\ldots.$ to $m$ factors$) \times (a.a.a\ldots$ to $n$ factors$)$ by multiplication.

or $a^m \times a^n = a.a.a.a\ldots\ldots\ldots\ldots$ to $\overline{m+n}$ factors $= a^{\overline{m+n}}$ by def.

$$\therefore\ a^m \times a^n = a^{\overline{m+n}}$$

*Note.*—The above proof will at first sight be difficult to understand. The pupil however must resolve to conquer the difficulty on account of the great importance of the rule. He will be assisted in his efforts if he will bear in mind that $m$ and $n$ are used as the general symbols of any positive whole number whatever; and that the above proof will be good if he suppose $m$ and $n$ to be some particular numbers. Thus if he were required to prove that $a^5 \times a^2 = a^7$, he would adapt the above general proof to that case by putting $m=5$; $n=2$. It would then read as follows.

For $a^5 = a.a.a.a.a$ to 5 factors by def.
And $a^2 = a.a$ to 2 factors by def.
Then $a^5 \times a^2 = (a.a.a.a.a$ to 5 factors$) \times (a.a..$ to 2 factors$)$
or $a^5 \times a^2 = a.a.a.a.a.a.a$ to 7 factors $a^5.a^2 = a^7$ by def.

And in a similar way he might write out the proof of any other particular case that might be proposed. He must also observe the great advantage of using general symbols ($m$ and $n$) instead of any particular numbers as 5, 2, &c. For when the proposition is proved with those particular numbers (say 5 and 2 as in the case just given) we do not prove that it is true for *all* other numbers, but since $m$ and $n$ may represent one number as well as another, there is in fact no possible case which is not included when those letters are used. The proof is called a *general* proof when $m$ and $n$ are used, but a *particular* proof when 5 and 2 are used.

# CHAPTER V.

## DIVISION.

**32.** *To divide one Algebraical Quantity by another is to find how many times the second is contained in the first, then to add the result if the divisor be positive, and to subtract it if the divisor be negative.*

We shall now consider four different cases of Division.

1°. To divide $12a$ by 4 is to find how often 4 is contained in $12a$, and add the result.

$$\therefore\ 12a \div 4 = 3a$$

2°. To divide $-12a$ by 4.

Now 4 is contained in $-12a$, $-3a$ times, for $-3a$ repeated 4 times gives $-12a$, and 4 repeated $-3a$ times must be the same thing.

$$\therefore\ -12a \div 4 = -3a$$

By similar reasoning it will appear that

3°. $$12a \div -4 = -3a$$

4°. $$-12a \div -4 = +3a$$

**33.** Hence the rule of signs in Division may be thus stated :—

*If the dividend and the divisor have the same sign, the quotient is positive, if they have different signs the quotient is negative.*

Ex. 1. Divide $14abc$ by $7a$.

Since $7a \times 2bc = 14abc$, it is evident that $14abc \div 7a = 2bc$.

Ex. 2. Divide $9a^6$ by $-9a^2$.

Since $-9a^2 \times -a^4 = 9a^6$, we have $9a^6 \div -9a^2 = -a^4$.

**33.** From the last two examples we may deduce the following Rule for the Division of two simple Algebraical Quantities.

### (*a*) RULE FOR THE SIGN OF THE QUOTIENT.

*When the dividend and divisor have each the same sign, the sign of the Quotient will be positive, but when the dividend and divisor have different signs, the sign of the Quotient will be negative.*

### (*b*) RULE FOR THE COEFFICIENT OF THE QUOTIENT.

*The Coefficient of the Quotient is found by dividing the Coefficient of the dividend by the Coefficient of the divisor.*

### (c) RULE FOR THE INDEX.*

*The Index of the Quotient is found by subtracting the Index of the divisor from the Index of the dividend.*

## Examples VIII.

(1.) Divide $12a^2$ by $2a$; $18ab$ by $3b$; $-15b^2$ by $b$; $25a^3$ by $-5a^2$; $-40a^3b^2$ by $-ab$; $a^3b^2c$ by $ab^2$; $-10x^3$ by $-2x^2$; $100x^4y^4$ by $xy^3$; $49a^5$ by $-7a^2$; $120abcxyz$ by $-12xyz$; $3a^2b^3c^4$ by $3ab^3c^4$.

*Note.*—If any of the coefficients be fractional the rules for division of fractions must be applied.

(2.) Divide $2x^2$ by $\frac{2}{3}x$

The coefficient of the quotient will be $2 \div \frac{2}{3} = 2 \times \frac{3}{2} = 3$

$$\therefore \quad 2x^2 \div \tfrac{2}{3}x = 3x$$

(3.) Find the following quotients :—

$10xy \div \frac{1}{5}x$; $2x^2 \div \frac{3}{4}x^2$; $17a^2b^2 \div -\frac{2}{17}a$; $\frac{3}{4}xyz \div 2xz$; $\frac{2}{9}x^4 \div \frac{1}{9}x^2$; $\frac{2}{3}a^2y^2 \div \frac{5}{2}y^2$; $ab \div \frac{1}{2}b$; $3b^2 \div \frac{1}{3}b$; $-x^4 \div 5x^2$; $-a^2x^2y^2z^2 \div 2az$; $-\frac{10}{15}ab^2 \div \frac{1}{3}b$; $-\frac{12}{13}c^2d^2 \div -20cd$; $-8x^2 \div -12x$; $-40a^2p^2q^2 \div -\frac{1}{60}aq^2$; $-\frac{24}{25}abcpqr \div -\frac{80}{17}acq$

(4.) Divide $a^2x^2+3abx^2-a^3x$ by $ax$

*Working.*
$$\begin{array}{r|l} ax & \underline{a^2x^2+3abx^2-a^3x} \\ & ax \quad +3bx \quad -a^2 \end{array}$$

The division in this case is effected by dividing each of the terms of the dividend by the divisor; the sum of all the separate quotients will be the complete quotient required.

Divide $6a^2+6b^2$ by $2$; $7a^2-14ab$ by $7a$; $x^2yz-xy^2z+xyz^2$ by $xyz$; $-4a^3+8a^2b-4a^2c$ by $-4a^2$; $3a^3-a^2b$ by $2a^2$; $3x^4-6x^3y+3x^2y^2-6xy^3$ by $12x$; $m^3+\frac{m^2n}{2}+\frac{mn^2}{3}$ by $2m$; $100m^2-10m+1$ by $10$; $\frac{3}{8}a^4-\frac{3}{4}a^3b+\frac{1}{2}b^4$ by $\frac{1}{2}b^2$; $\frac{1}{14}p^3+\frac{2}{7}p^2q$ by $\frac{2}{7}p$; $a^4+3a^3x+a^2x^2$ by $2a^2$; $\frac{1}{4}x^3yz-\frac{1}{6}xy^3z+\frac{1}{4}xyz^3$ by $\frac{1}{16}xyz$; $9x^3yz-x^3yz+\frac{1}{3}xyz^3$ by $\frac{1}{3}xz$

(5.) Divide $a^2-b^2$ by $a-b$

In this case the process of division is more complicated, and the method to be pursued is similar to that of Long Division in Arithmetic, as will be seen in the following working :—

$$\begin{array}{l} a-b)\ a^2-b^2\ (a+b \\ \phantom{a-b)\ }\underline{a^2-ab} \\ \phantom{a-b)\ a^2-{}}ab-b^2 \\ \phantom{a-b)\ a^2-{}}\underline{ab-b^2} \\ \phantom{a-b)\ a^2-{}}\quad \cdot\ \cdot \end{array}$$

---

* See p. 28.

*Note.*—The rules to be observed in the division of quantities consisting of more than one term are 1°. To arrange both the divisor and dividend according to the powers of the same letter ($a$ in the example); then to divide the first term of the dividend by the first term of the divisor, place the result in the quotient and multiply the divisor by it; subtract and proceed similarly with the next line, and so on to the end.

(6.) Divide $a^3-6a^2+12a-8$ by $a-2$

$$\begin{array}{l}
a-2)\,a^3-6a^2+12a-8\,(a^2-4a+4 \\
\qquad\quad a^3-2a^2 \\
\qquad\quad \overline{\qquad\quad -4a^2+12a} \\
\qquad\qquad\quad -4a^2+8a \\
\qquad\qquad\quad \overline{\qquad\qquad 4a-8} \\
\qquad\qquad\qquad\qquad 4a-8 \\
\qquad\qquad\qquad\qquad \overline{\qquad\quad}
\end{array}$$

(7.) Divide $4x-x^2+x^4-4$ by $x^2-2+x$

Here the dividend and divisor must be arranged according to powers of $x$, as follows:—

$$\begin{array}{l}
x^2+x-2)\,x^4-x^2+4x-4\,(x^2-x+2 \\
\qquad\qquad\quad x^4+x^3-2x^2 \\
\qquad\qquad\quad \overline{\qquad -x^3+x^2+4x} \\
\qquad\qquad\qquad -x^3-x^2+2x \\
\qquad\qquad\qquad \overline{\qquad\qquad 2x^2+2x-4} \\
\qquad\qquad\qquad\qquad\quad 2x^2+2x-4 \\
\qquad\qquad\qquad\qquad\quad \overline{\qquad\qquad}
\end{array}$$

(8.) Divide $x^2+x-2$ by $x+2$; and divide $3x^3+3x^2-x-1$ by $3x^2-1$

(9.) Divide $4a^3-a$ by $2a-1$; $x^2-16$ by $x-4$

(10.) Divide $x^4-1$ by $x-1$; $x^3+8$ by $x+2$

(11.) Divide $3x^4-6x^3-2x^2+5x-12$ by $3x^2-3x+4$

(12.) Divide $14x^4-65x^3+141x^2-116x+44$ by $2x^2-7x+11$

(13.) Divide $a^3-8b^3$ by $a^2+2ab+4b^2$

(14.) Divide $a^5+32$ by $a^4-2a^3+4a^2-8a+16$

(15.) Divide $a^4x^4-y^4$ by $ax-y$; $100a^2-b^2$ by $10a+b$

(16.) Divide $x^4+x^2y^2+y^4$ by $x^2-xy+y^2$

(17.) Divide $3x^2y^2z^2+14xyz-5$ by $3xyz-1$

(18.) Divide $6a^2b^2-abcd-12c^2d^2$ by $3ab+4cd$

(19.) Divide $8a^3-27$ by $9-12a+4a^2$

(20.) Divide $a^2+2ab+b^2-c^2$ by $a+b+c$; by $a+b-c$; by $b+c-a$; and by $a+c-b$

(21.) Divide $a^3-a^2b-a^2c+ab^2+ac^2-b^3-bc^2-b^2c-c^3$ by $a^2+b^2+c^2$

(22.) Divide $8a^4-16a^3+12a^2$ by $4a^2$

(23.) Divide $6x^2yz-12xy^2z+18xyz^2$ by $6xyz$

(24.) Divide $a^6-p^6q^6$ by $a^5+a^4pq+a^3p^2q^2+a^2p^3q^3+ap^4q^4+p^5q^5$

(25.) Divide $3p^3-4p^2q+3p^2r-7pq^2-7pqr+12qr^2+12r^3+12pr^2$ by $3p^2-7pq+12r^2$

(26.) Divide $x^3-1$ by $x+1$

*Working.*

$$\begin{array}{l} x+1)\,x^3-1\,(x^2-x+1 \\ \quad\quad\;\; x^3+x^2 \\ \quad\quad\;\; \overline{\quad -x^2-1} \\ \quad\quad\quad\;\; -x^2-x \\ \quad\quad\quad\;\; \overline{\qquad\quad x-1} \\ \qquad\qquad\quad\;\; x+1 \\ \qquad\qquad\quad\;\; \overline{\quad\;\, -2} \end{array}$$

The division in this case does not terminate; the quotient is $x^2-x+1$; and there is a remainder $-2$. The proper way to write the answer is—

$$x^2-x+1-\frac{2}{x+1}$$

(27.) Divide $2a$ by $1-x^2$

*Working.*

$$\begin{array}{l} 1-x^2)\,2a\qquad(2a+2ax^2+2ax^4+2ax^6 \\ \qquad\;\; 2a-2ax^2 \\ \qquad\;\; \overline{\qquad 2ax^2} \\ \qquad\qquad\; 2ax^2-2ax^4 \\ \qquad\qquad\; \overline{\qquad\quad 2ax^4} \\ \qquad\qquad\qquad\;\; 2ax^4-2ax^6 \\ \qquad\qquad\qquad\;\; \overline{\qquad\quad 2ax^6} \\ \qquad\qquad\qquad\qquad\;\; 2ax^6-2ax^8 \\ \qquad\qquad\qquad\qquad\;\; \overline{\qquad\quad 2ax^8} \end{array}$$

Here it is plain that the division will never terminate. In fact the quotient is an endless series or terms, four of which have been found; it is easily seen that the next 4 terms are—

$$2ax^8,\ 2ax^{10},\ 2ax^{12},\ 2ax^{14}.$$

When the division is carried on to 4 terms only, the result is—

$$2a+2ax^2+2ax^4+2ax^6+\frac{2ax^8}{1-x^2}$$

(28.) Divide $x$ by $1+x$ to 4 terms.

(29.) Divide 3 by $1-x$; $6+a$ by $1-a$, both to 4 terms.

(30.) Divide $a^2x^2$ by $a-ax$ to 6 terms.

(31.) Divide $1+x^2$ by $1-x+x^2$ to 4 terms.

(32.) Find the quotient to 4 terms of each of the following divisions:—
$1+x^3\div 1-x^3$; $x^7\div x-1$; $a^6\div a-3$; $1\div 1+a$; $(x+a)^4\div x-a$; $(x-a)^4\div x-a$; $(1+x)^3\div(1-x)^2$; $x^6+x^3+1\div x^2+x+1$.

) Divide $x^2-\frac{1}{2}x-\frac{1}{2}$ by $x-1$

*Working.* $x-1)\,x^2-\frac{1}{2}x-\frac{1}{2}\,(x+\frac{1}{2}$

$x^2-x$

$\frac{1}{2}x-\frac{1}{2}$

$\frac{1}{2}x-\frac{1}{2}$

$\cdot\quad\cdot$

) Divide $\frac{1}{3}a^4+\frac{1}{2}a^3b-\frac{7}{9}a^2b^2+\frac{1}{3}ab^3-\frac{2}{3}b^4$ by $\frac{1}{3}a^2+\frac{1}{2}ab-b^2$

$\frac{1}{3}a^2+\frac{1}{2}ab-b^2)\,\frac{1}{3}a^4+\frac{1}{2}a^3b-\frac{7}{9}a^2b^2+\frac{1}{3}ab^3-\frac{2}{3}b^4\,(a^2+\frac{2}{3}b^2$

$\frac{1}{3}a^4+\frac{1}{2}a^3b-a^2b^2$

$\frac{2}{9}a^2b^2+\frac{1}{3}ab^3-\frac{2}{3}b^4$

$\frac{2}{9}a^2b^2+\frac{1}{3}ab^3-\frac{2}{3}b^4$

n Ex. 33 and 34, it will be observed that the fractional coefficients t be added, subtracted, multiplied and divided by the Rules for gar fractions. In other respects the working is the same as for le numbers.

) Divide $x^2-\frac{1}{4}$ by $x-\frac{1}{2}$; $x^2-\frac{1}{9}$ by $x+\frac{1}{3}$; $a^2-\frac{5}{6}a+\frac{1}{6}$ by $a-\frac{1}{2}$; $a^2+ax$ by $\frac{2}{3}a$; $3m^3-m^2+\frac{m}{3}$ by $2m$; $\frac{1}{4}a^4-1$ by $\frac{1}{2}a^2+1$; $\frac{1}{8}a^3+b^3$ by $\frac{1}{4}a^2-\frac{1}{2}ab+b^2$.

) Find the following quotients :—

$3a^3-3$ by $\frac{1}{2}a^2+\frac{1}{2}a+\frac{1}{2}$; $\frac{1}{4}x^2-\frac{1}{2}xy+\frac{1}{4}y^2\div\frac{1}{2}x-\frac{1}{2}y$;

$a^5-\frac{1}{243}\div a-3$; $a^4-\frac{16}{81}\div a^3+\frac{2}{3}a^2+\frac{4}{9}a+\frac{8}{27}$;

$\frac{4}{9}p^3q-2p^2q^2+\frac{9}{4}pq^3\div\frac{2}{3}p-\frac{3}{2}q$

$\frac{a^4}{4}-\frac{3}{8}a^3b+\frac{13}{16}a^2b^2+\frac{3}{8}ab^3-\frac{b^4}{4}$ by $\frac{a^2}{4}-\frac{ab}{2}+b^2$

.) Divide 1 by $x+\frac{1}{3}$; $x+\frac{1}{2}$ by $\frac{1}{3}+x$; $1+x^2$ by $1+\frac{x}{3}$; $1+\frac{1}{2}x$ by $1-\frac{1}{2}x^2$ each to four terms.

.) Divide $x^2+(a+b)x+ab$ by $x+b$

*Working.* $x+b)\,x^2+ax+bx+ab\,(x+a$

$x^2+bx$

$ax+ab$

$ax+ab$

.) Divide $x^3-(a+b+c)\,x^2+(ab+ac+bc)\,x-abc$ by $x-c$

$x-c)\,x^3-(a+b+c)x^2+(ab+ac+bc)x-abc\,(x^2-(a+b)x+4b$

$x^3-cx^2$

$-(a+b)x^2+(ab+ac+bc)x$

$-(a+b)x^2+(ac+bc)x$

$abx-abc$

$abx-abc$

*Note.*—In working examples like (39) the easiest way is to keep the coefficients of the different powers of $x$ in brackets throughout; the rules for the addition and subtraction of bracketed quantities being carefully observed.

(40.) Divide $x^2+2(a-1)x+a^2-2a+1$ by $x+a-1$.
(41.) Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.
(42.) Divide $a^2-b^2-2bc-c^2$ by $a-b-c$.
(43.) Divide $x^3-ax^2-(a+2)x+a^2-1$ by $x+1$.
(44.) Divide $a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2$ by $a^2+2ab+b^2-c^2$.
(45.) Divide $x^4-(a^2+2ab+b^2)x^2$ by $x^2-(a+b)x+ab$.
(46.) Divide $a^4+ab^2p-(2a^2+ab+b^2)p^2+bp^3+p^4$ by $a^3+(a^2+b^2)p-(a+b)p^2-p^3$.
(47.) Divide $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$, separately, by $b+b-a$, $a+b-e$, $a+c-b$, $a+b+c$.

### PROOF OF RULE (*c*), p. 24.

**35.** The Index of the Quotient is found by subtracting the Index of the Divisor from the Index of the Dividend.

Or, $a^m \div a^n = a^{m-n}$

For $a^m = a.a.a.......$ to $m$ factors by Def.

And $a^n = a.a.a.......$ to $n$ factors by Def.

$\therefore a^m \div a^n = (a.a.a...$ to $m$ factors$) \div (a.a\ a...)$ to $n$ factors

$$= \frac{a.a\ a...\text{ to } m \text{ factors}}{a.a.a...\text{ to } n \text{ factors}}$$

$$= \frac{a.a...\text{ to } \overline{m-n} \text{ factors}}{1}$$

By cancelling all the $n$ factors common to the numerator and denominator

$$= \overline{a^{m-n}} \text{ by Definition}$$

$$\therefore a^m \div a^n = \overline{a^{m-n}}$$

$m$ being greater than $n$.

If $n$ be greater than $m$, then the $m$ factors common to the numerator and denominator must be cancelled, and the result will be

$$= \frac{1}{a^{n-m}}$$

Or $$a^m \div a^n = \frac{1}{a^{n-m}}$$

If $m$ be less than $n$.

## Exercise on Chapter V.

(1.) Divide by $(a-b)$ the following quantities :—

$a^2-b^2$ ; $a^3-b^3$ ; $a^4-b^4$ ; $a^5-b^5$ ; $a^6-b^6$.

Hence it may be seen that $a^m-b^m$ is always divisible by $(a-b)$ without remainder, whatever positive whole number $m$ may be.

(2.) Divide by $(a+b)$ the following quantities :—

$a^2-b^2$; $a^3-b^3$; $a^4-b^4$; $a^5-b^5$; $a^6-b^6$; $a^7-b^7$.

From these results it may be seen that $a^m-b^m$ is divisible by $(a+b)$ when $m$ is *even*, but not when $m$ is odd.

(3.) Divide by $(a-b)$ the following quantities :—

$a^2+b^2$; $a^3+b^3$; $a^4+b^4$; $a^5+b^5$; $a^6+b^6$; $a^7+b^7$.

Since remainders occur in all these cases, it may be seen that $a^m+b^m$ is never divisible by $(a-b)$ whatever whole number $m$ may be.

(4.) Divide by $(a+b)$ the following quantities :—

$a^2+b^2$ ; $a^3+b^3$; $a^4+b^4$; $a^5+b^5$; $a^6+b^6$ ; $a^7+b^7$.

From these results it may be seen that $a^m+b^m$ is divisible by $a+b$ when $m$ is *odd*, but not when $m$ is even.

(5.) Taking as an example $a^8-b^8 \div a-b$, we have $a^8-b^8 \div a-b = a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b7$

The form of this quotient should be noticed.

(a) The number of terms is 8, the same as the index of $a$ and $b$.

(b) The highest power of $a$ comes first
The highest power of $b$ comes last
The powers of $a$ follow in descending order
The powers of $b$ follow in ascending order

(c) All the signs are positive.

Take an example of the kind in (2), (3), (4) respectively; write down its quotient, and draw conclusions as to the indices and signs similar to (but not the same as) (a), (b), (c).

(6.) Write down, without working the division, the quotients of—

$x^3-y^3 \div x-y$
$x^6-y^6 \div x-y$
$x^7-y^7 \div x-y$
$x^9-y^9 \div x-y$
$x^{11}+y^{11} \div x+y$.
$x^4-y^4 \div x+y$
$x^5+y^5 \div x+y$
$x^7+y^7 \div x+y$
$x^9+y^9 \div x+y$
$x^3+y^3 \div x+y$
$x^5-y^5 \div x-y$
$x^8-y^8 \div x-y$
$x^{15}-y^{15} \div x-y$

(7.) Divide $27y^3-1$ by $3y-1$

*Working.* $27y^3-1 \div 3y-1 = (3y)^3-(1)^3 \div 3y-1$
$= (3y)^2+(3y)\ 1+(1)^2 = 9y^2+3y+1$

Similarly. $32a^5+243b^5c^{10}\div 2a+3bc^2$
$=(2a)^5+(3bc^2)^5\div 2a+3bc^2$
$=(2a)^4-(2a)^3.(3bc^2)+(2a)^2.(3bc^2)^2-(2a).(3bc^2)^3+(3bc^2)^4$
$=16a^4-24a^3bc^2+36a^2b^2c^4-54ab^3c^6+81b^4c^8$

In like manner find the following quotients;—

$4a^2-1\div 2a+1$; $1+8a^3\div 1+2a$
$1-8a^3\div 1-2a$; $16a^2-1\div 4a-1$
$9a^6-b^4\div 3a^3+b^2$; $4a^2b^2-c^2\div 2ab-c$
$25a^2b^2c^2-16x^2\div 5abc+4x$; $a^4-1\div a-1$
$a^4-1\div a^2-1$; $a^4-b^4\div a^2+b^2$
$9a^2-16b^2\div 3a-4b$; $16x^4y^4-z^4\div 4xy+z$
$27a^3-8b^3c^3\div 3a-2bc$; $81a^4-1\div 3a+1$
$343a^3-8\div 7a-2$; $243a^5+32b^5\div 3a+2b$
$64a^3d^3+8b^6c^6\div 4ad+2b^2c^2$; $32x^5y^5z^5+1024a^5b^5c^5\div 2xyz+4abc$.

# CHAPTER VI.

## INVOLUTION AND EVOLUTION;

Or, the Method of obtaining the Powers and Roots of Algebraical Quantities.

**37.** *Involution is the method of raising an Algebraical Quantity to any required power.*

By Def. 10 it appears that this process consists in successive multiplications of the same factor.

Thus the square or 2nd power of 3
$$=3\times 3=9$$

The cube or 3rd power of $2a$
$$=2a\times 2a\times 2a$$
$$=8a^3.$$

We shall now consider some particular cases of Involution to which short and convenient rules may be applied.

Ex. (1.) Raise $a^2$ to the third power.

Since $$(a^2)^3=a^2\times a^2\times a^2$$
$$=a^4\times a^2=a^6$$
$$\therefore\ (a^2)^3=a^6.$$

Similarly it will appear that
$$(a^2)^5=a^{10}$$

Ex. (2.) Find the 4th power of $2ab^2c^3$.

$$\begin{aligned}(2ab^2c^3)^4 &= 2ab^2c^3 \times 2ab^2c^3 \times 2ab^2c^3 \times 2ab^2c^3\\ &= 2 \times 2 \times 2 \times 2 \times a.a.a.a.b^2.b^2.b^2.b^2.c^3.c^3.c^3.c^3\\ &= 2^4.a^4.b^8.c^{12}\end{aligned}$$

or $$(2ab^2c^3)^4 = 15a^4b^8c^{12}.$$

**38.** From these examples we may deduce the following *Rule for raising a quantity of one term to any required power.*

RULE.

*Multiply the index of each letter by the index of the required power.*

*Note.*—This rule is in fact the same as that given in p. 19 for the index of the product of different powers of the same letter; for by that rule we *add* the indices of the factors together; but, in the case of Involution, the indices of all the factors are equal, and to *add* a number of *equal* indices together is the same thing as to multiply one of the indices by the number of them.

Ex. (3.) Raise $-a^2xy$ to the third power.

$$\begin{aligned}(-a^2xy)^3 &= (-a^2xy).(-a^2xy).(-a^2xy)\\ &= (a^4x^2y^2).(-a^2xy)\\ &= -a^6x^3y^3.\end{aligned}$$

Hence it appears that when the quantity is negative and the index of the required power is odd, the sign of the required power is also negative; for the product of an *odd* number of negative factors must be *negative*.

But, if the quantity be negative, and the index of the required power be even, then the sign of the required power is positive; for the product of an *even* number of negative factors must be *positive*.

Ex. (4.) Find the square of $(a+b)$.

By the accompanying multiplication it is seen that

$$\begin{array}{l} a+b\\ a+b\\ \hline a^2+ab\\ \quad +ab+b^2\\ \hline a^2+2ab+b^2\\ \hline \end{array}$$

$$(a+b)^3 = a^2+2ab+b^2$$

Similarly we shall find by multiplication, that

$$(a-b)^2 = a^2 - 2ab + b^2$$

Now $a^2+2ab+b^2$ is made up of the square of $a$, the square of $b$, and twice the product of $a$ and $b$.

And $a^2-2ab+b^2$ is made up of the square of $a$, the square of $(-b)$, and twice the product of $a$ and $(-b)$.

Or, the square of any binomial $(a+b)$ consists of three terms

1°. The square of $a$; $a^2$
2°. The square of $a$; $b^2$
3°. Twice the product of the two quantities $a$ and $b$; $2ab$.

The learner is requested to pay very particular attention to this Rule; for more errors arise from the neglect of it than from any other cause.

Ex. (5.) Find the square of $3a-2b$.

$$(3a-2b)^2=(3a)^2+2.(3a).(-2b)+(-2b)^2$$
$$=9a^2-12ab+4b^2.$$

### Examples IX.

Ex. (6.) Write down the squares of the following quantities:—

$3ab$, $8a^2c$, $-9a^3$, $-ax^2y^3$, $-10az^2$, $8p^2q^3$, $2a$, $4x^{10}y$, $-2x^2y^2z^2$, $axy$, $-bc^2d^2$, $-4x^2y^3z^4$.

Ex. (7.) Obtain the following powers:—

$(-8a)^3$, $(4a^2)^2$, $(7ab)^3$, $(-2b^3)^4$, $(-3abc)^3$, $(3x^2)^4$, $(-xyz)^5$, $(2x^2y^4z^6)^4$, $(10p^3q^3r^3)^3$, $(-100a^5)^2$.

Ex. (8.) Write down the squares of the following quantities:—

$(a-1)$, $a+2$, $3a+2b$, $7a+4$, $5x-2y$, $ax-by$, $7z-4$, $abc-4$, $ax^2+by^2$, $3x^3-4y^3$, $x^2yz-2b^2c$, $a^3x^3+b^3y^3$, $bx^3-ay^3$, $-a+1$, $-xy-yz$.

To find the square of $a+b+c$,

$$(a+b+c)^2=(\overline{a+b}+c)^2$$
$$=\overline{a+b}|^2+2\overline{a+b}.c+c^2$$

if $\overline{a+b}$ be regarded as a single quantity.

$$\therefore\ (a+b+c)^2=(a^2+2ab+b^2)+2ac+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc$$

Similarly $(a+b-c)^2=a^2+b^2+c^2+2ab-2ac-2bc$.

Again, the square of $a+b-c+d$, may be found in a similar manner.

For
$$(a+b-c+d)^2=(\overline{a+b-c+d})^2$$
$$=(\overline{a+b-c})^2+2\overline{a+b-c}.d+d^2$$
$$=a^2+b^2+c^2+2ab-2ac-2bc+2ad+2bd-2cd+d^2$$
$$=a^2+b^2+c^2+d^2+2ab-2ac+2ad-2bc+2bd-2cd.$$

**39.** By observing the two last results we may deduce the following *Rule* for obtaining the square of any Multinomial.

### RULE.

*Write down the square of each term, and then twice the product of each term into the several terms that follow it.*

## Examples X.

Find the square of $2a-3b+c$.

By the Rule $(2a-3b+c)^2=4a^2+9b^2+c^2-12ab+4ac-6bc$.

(2.) Write down the squares of the following quantities :—

$(a+b-c-d)$, $(a-2b+3c)$, $(1-x+x^2)$, $(1+x-x^2)$
$(a-2b-3c-3d)$, $(1-2x+3x^2-3x^3)$, $(a-2b+3c-4d+5e)$
$(abc+abd+acd+bcd)$, $(1-2x+3x^2-3x^3+4x^4-5x^5)$
$2x^2-3x+7$, $x^2-7x+12$, $a^2+ab+b^2$.

(3.) Expand the following expressions :—

$(1-x)^4$, $(1-x+x^2)^4$, $(1-x)^3$, $\left(\frac{a^2}{2}+1\right)$, $\left(\frac{a}{2}-\frac{b}{3}\right)^2$

$\left(1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}\right)^2$, $(2a-5b)^3$, $(ax^2-bx+c)^2$, $(3x-1)^4$

$(x^2+xy+y^2)^4$, $\left(\frac{ax}{2}-\frac{by}{4}\right)^4$, $\left(x^3-\frac{x^2}{2}+\frac{x}{4}-\frac{1}{8}\right)^4$.

$(1+x+x^2+x^3)^3$, $(a-b+c-d)^3$, $1-2x+3x^2-4x^3)^3$

$\left(\frac{a^2}{2}-\frac{x^2}{3}\right)^5$, $\left(1-\frac{x^4}{2}\right)^5$, $(a+bx)^7$.

(4.) Let it be required to find the coefficient of $x^3$ in the expansion of

$$(1+ax+bx^2+cx^3)^2$$

Since we know that the square of the whole quantity within the brackets consists of the squares of each term, together with twice the product of each term into every term that follows it, we need write down only those terms which produce $x^3$. Now the only products which contain $x^3$, are $1\times cx^3$ and $ax\times bx^2$,

Therefore the coefficient of $x^3$ in the expansion is $2c+2ab$.

Similarly, the following may be found :—

The coefficient of $a^4$ in $(a^4+a^3+a^2+1)^2$
Of $x^5$ in $(1-x+x^2)^4$, of $a^3b$ in $(a^2-ab+b^2)^2$
Of $y^3$ in $(1-y)^4$, of $x^2$ in $(x^2+xy+x^2)^3$.

**40.** Evolution is the method of extracting any required root of a given quantity.

Since $x^6 \times x^6 = x^{12}$

$\therefore \sqrt{x^{12}} = x^2$ by Definition.

Also, $x^4 \times x^4 \times x^4 = x^{12}$

$\therefore \sqrt[3]{x^{12}} = x^4$ ,,

$x^3 \times x^3 \times x^3 \times x^3 = x^{12}$

$\therefore \sqrt[4]{x^{12}} = x^3$ ,,

$x^2 . x^2 . x^2 . x^2 . x^2 . x^2 = x^{12}$

$\therefore \sqrt[6]{x^{12}} = x^2$ ,,

Since $2x \times 2x \times 2x = 8x^3 = 2^3 x^3$

$\therefore \sqrt[3]{8x^3} = 2x$

And, $3a^2 . 3a^2 . 3a^2 . 3a^2 = 81a^8 = 3^4 . (a^2)^4$

$\therefore \sqrt[4]{81a^8} = 3a^2$

Again, $4a^2bc . 4a^2bc . 4a^2bc = 64a^6b^3c^3 = 4^3a^6b^3c^3$

$\therefore \sqrt[3]{64a^6b^3c^3} = 4a^2bc.$

**41.** From these examples we may deduce the following *Rule for extracting any required root of a given quantity consisting of a single term.*

RULE.

*Express the numerical coefficient as the power of a number, having the same index as the required root. Then divide each index in the quantity by the index of the required root.*

### Examples XI.

(1.) Find the square root of $a^2b^2$, of $4x^4$, of $9a^2b^2x^6$

*Working.* $\sqrt{a^2b^2} = a^1b^1$ by dividing each index by 2

or $\sqrt{a^2b^2} = ab$ the figures in this case being omitted.

Also, $\sqrt{4x^4} = \sqrt{2^2x^4} = 2^1 . x^2$
$= 2x^2$

And, $\sqrt[3]{27a^3b^3x^6} = \sqrt[3]{3^3a^3b^3x^6} = 3^1 . a^1 . b^1 . x^2$
$= 3abx^2.$

(2.) Find the square root of $16a^4$, $9x^2$, $25m^2n^2p^2$, $36a^4b^8c^{10}$, $49a^2x^4$, $64a^6b^{12}$, $a^6b^8c^2d^{10}$.

(3.) Find the cube root of $a^3$, $8a^3$, $8a^3b^3c^3$, $8a^6b^4c^2$, $125a^3$, $216x^3y^6z^9$, $343a^{12}b^{12}c^{24}$, $512m^3n^6p^3$, $729z^{15}$, $27m^3n^3$, $64x^3y^9$.

(4.) Find the 4th root of $16a^4b^4$, $81a^4b^8c^{12}$, $x^4y^4z^8$.

(5.) Express the following in their simplest forms :—

$$\sqrt[3]{27a^3},\ \sqrt[3]{a^6x^6},\ \sqrt[4]{256x^4y^8},\ \sqrt[5]{32a^5b^{10}c^{15}},\ \sqrt[6]{1}.\ \sqrt[7]{1},\ \sqrt[6]{64a^6},$$
$$\sqrt[7]{128a^{14}},\ \sqrt[3]{27m^3n^{12}},\ \sqrt{49a^4x^2},\ \sqrt{121x^2y^6z^{10}}.$$

**42.** Since $(+a)\times(+a)=+a^2$

and $(-a)\times(-a)=+a^2$

$$\therefore\ \left.\begin{array}{l}\sqrt{a^2}=+a\\ \text{and}\ \sqrt{a^2}=-a\end{array}\right\}$$

Similarly $\left.\begin{array}{l}\sqrt[4]{a^8}=+a^2\\ \text{or}\quad -a^2\end{array}\right\}$

and thus it may be made to appear generally, that an *even* root of a *positive* algebraical quantity is either *positive* or *negative*.

Again, since $(+a)\times(+a)\times(+a)=+a^3$

and $(-a)\times(-a)\times(-a)=-a^3$

$$\therefore\ \left.\begin{array}{l}\sqrt[3]{a^3}=+a\\ \sqrt[3]{-a^3}=-a\end{array}\right\}$$

Hence it may be shown that every *odd* root of a positive quantity is only positive, and every *odd* root of a negative quantity is only negative.

Also, it is impossible to extract any *even* root of a *negative* quantity; for any even number of equal factors multiplied together must give a positive result.

(6.) Extract the square root of $\frac{1}{4}a^2b^2$.

Since $\sqrt{\frac{1}{4}}=\frac{1}{2}$

$$\therefore\ \sqrt{\tfrac{1}{4}a^2b^2}=\tfrac{1}{2}ab.$$

(7.) Find the square roots of the following quantities :—

$$\tfrac{1}{9}a^2,\ \frac{9a^2}{16b^2},\ \frac{1}{25a^2b^2},\ \frac{9x^2}{100},\ \frac{16x^2y^4z^6}{81a^2b^4c^6}.$$

(8.) Express the following in their simplest forms :—

$$\sqrt[3]{\frac{1800}{27}a^6},\ \sqrt[3]{3\tfrac{3}{8}\times a^6b^9},\ \sqrt[3]{\frac{64\ x^6y^{12}}{27\ a^{12}b^6}}$$
$$\sqrt[4]{\frac{x^8}{81}},\ \sqrt[5]{\frac{32a^5b^5c^5}{243a^{10}b^{10}c^{10}}},\ \sqrt{\frac{169a^6b^{10}c^2}{196a^4b^{10}c^4}}$$

**43.** To investigate a method of extracting the square root of any compound quantity.

Since $(a+b)\,.\,(a+b)=a^2+2ab+b^2$

$\therefore\ \sqrt{a^2+2ab+b^2}=a+b$, by Def. of the square root.

We must now consider, how $(a+b)$ might be deduced from $a^2+2ab+b^2$, the latter quantity being given.

The first term $(a)$ of the root is evidently the square root of $(a^2)$, that is, the first term of the root is obtained by taking the square root of the first term of the given quantity.

$$\begin{array}{rl} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & \cdot\quad\cdot \\ \hline \end{array}$$

If, then, $a^2$ be subtracted from the whole quantity, there remains $2ab+b^2$, and to obtain $b$ from this, it is necessary to divide the first of its terms by $(2a)$, the double of the term already obtained: this result is $(b)$; and if $2a+b$ be multiplied by $(b)$, the product is $2ab+b^2$, which just makes up the remaining quantity.

**The Rule for extracting the square root of a quantity may be stated as follows.**

(a.) *Arrange the terms according to the descending powers of the same letter.*

(b.) *Extract the square root of the first term and write it in the quotient, square this, and subtract the result from the given quantity, and consider the remainder as a new dividend.*

(c.) *Double the quotient and write it as a trial divisor; find how often this is contained in the first term of the dividend, and write this result both in the quotient and divisor.*

(d.) *Multiply the whole of the divisor thus formed by the term last written in the quotient, and subtract the result.*

(e.) *If there be still a remainder, proceed again according to* (c.), *remembering to double the whole of the quotient.*

## Examples XII.

(1.) Find the square roots of the following quantities :—

$a^2-4ab+4b^2$

$x^2-6x+9$

$x^4+2x^2+1$

$a^6-10a^3+25$

$9a^8-24a^4b+16b^2$

$16a^2b^2+40abcd+25c^2d^2$

$144x^2z^4-72x^2y^2z^2+9x^2y^4$

$4a^2+4ab+b^2$

$4x^2-12x+9$

$a^2x^2-4ax+4$

$4a^4b^2+28a^2b+49$

$25x^4+60x^2y+36y^2$

$36x^6y^4z^2-84x^3y^2z+49$

$49a^2d^2-154abcd+121b^2c^2$.

(2.) Find the square root of $a^6-4a^5+8a^4-10a^3+8a^2-4a+1$.

*Working.*

$$\begin{array}{rl}
 & a^6-4a^5+8a^4-10a^3+8a^2-4a+1\ (a^3-2a^2+2a-1 \\
 & a^6 \\ \hline
2a^3-2a^2) & -4a^5+8a^4 \\
 & -4a^5+4a^4 \\ \hline
2a^3-4a^2+2a) & 4a^4-10a^3+8a^2 \\
 & 4a^4-\ 8a^3+4a^2 \\ \hline
2a^3-4a^2+4a-1) & -\ 2a^3+4a^2-4a+1 \\
 & -\ 2a^3+4a^2-4a+1 \\ \hline
 & \cdot\quad\cdot\quad\cdot\quad\cdot \\ \hline
\end{array}$$

*Ans.* $a^3-2a^2+2a-1$.

(3.) Find the square roots of the following :—

$a^2+b^2+c^2+2ab-2ac-2bc$

$a^2+4b^2+9c^2-4ab-6ac+12bc$

$9x^6-18x^4-6x^3+9x^2+6x+1$

$9a^6-12a^5+46a^4+38a^3+5a^2+154a+121$

$x^{12}-2x^{10}y^2+3x^8y^4-4x^6y^6+3x^4y^8-2x^2y^{10}+y^{12}$.

(4.) Find the square roots of

$a^2-ab+\frac{b^2}{4}$ ; $4a^4-\frac{4}{3}a^3+\frac{a^2}{9}$

$\frac{4}{25}x^2-\frac{8}{5}x+1$ ; $\frac{4}{9}x^2+2xy+\frac{9}{4}y^2$

$\frac{x^4}{4}+\frac{x^2y^2}{16}+\frac{y^4}{4}-\frac{x^3y}{4}+\frac{x^2x^2}{2}-\frac{xy^3}{4}$

$4a^4b^2-2a^3b^3+\frac{a^2b^4}{4}$

$\frac{1}{4}a^4+ba^3+\frac{4}{3}a^2b^2+\frac{2}{3}ab^3+\frac{1}{9}b^4$.

(5.) Find $\sqrt{x^{2m}-2x^m+1}$

*Working.*

$$\begin{array}{rl}
 & x^{2m}-2x^m+1(x^m-1 \\
 & x^{2m} \\ \hline
2x^m-1) & -2x^m+1 \\
 & -2x^m+1 \\ \hline
 & \cdot\quad\cdot \\ \hline
\end{array}$$

*Ans.* $x^m-1$.

Where the indices contain letters, as in this example, the same rules must be observed as when they are whole numbers.

(6.) Find the square roots of the following :—

$4a^{2m}-12a^m+9$ ; $a^{2p}-4a^pb^q-4b^{2q}$

$a^{4p}-6a^{3p}+15a^{2p}-18a^p+9$

$9a^{2n}b^{2m}-24a^{\overline{m+n}}.\,b^{\overline{m+n}}+16a^{2m}b^{2n}$

$$x^{2m}y^4-8x^{\overline{m+2}}.y^{\overline{n+2}}+16x^4y^{2n}.$$

$$\frac{4x^2y^4}{25}-\frac{4x^3y^3}{3}+\frac{179x^4y^2}{45}-5x^5y+\frac{9x^6}{4}.$$

**44.** We shall now investigate a method of extracting the cube root of a given quantity.

By actual multiplication it is seen that

$$(a+b).(a+b).(a+b)=a^3+3a^2b+3ab^2+b^3;$$

that is, the cube root of $a^3+3a^2b+3ab^2+b^3=a+b$, and we must now consider how $(a+b)$ may be found from $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\ \hline
3a^2+3ab+b^2) & \quad 3a^2b+3ab^2+b^3 \\
 & \quad 3a^2b+3ab^2+b^3 \\ \hline
 & \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

Now $a$, the first term of the root, is the cube root of the first term $a^3$. Let $a$ be written in the quotient; then, if $a$ be cubed and subtracted from the original quantity, there remains $3a^2b+3ab^2+b^3$, and $b$ may be obtained from the first term of this by dividing it by $3a^2$. Let $3a^2$ be written as the beginning of a new divisor, and $b$ be written in the quotient. Then, if the two terms $3ab+b^2$ be annexed to $3a^2$ in the divisor, and these three terms be multiplied by $b$, the result will be $3a^2b+3ab^2+b^3$, which, being subtracted, leaves no remainder.

Hence, the Rule for extracting the cube root of a compound quantity may be stated as follows.

(a.) *Arrange the terms according to the descending powers of the same letter* $(a)$.

(b.) *Extract the cube root of the first term; write this root* $(a)$ *in the quotient, cube it and subtract. Consider the remainder as a new dividend.* $(3a^2b+3ab^2+b^3)$.

(c.) *Begin to form a new divisor by squaring the quotient* $(a)$ *and multiplying it by* 3 $(3a^2)$.

(d.) *Divide the first term of the dividend by this trial divisor, and write the result* $(b)$ *in the quotient.*

(e.) *Complete the divisor by annexing to it three times the product of the two terms of the quotient* $(3ab)$, *and the square of its last term* $(b)$.

(f.) *Multiply the divisor* $(3a^2+3ab+b^2)$ *by the last term* $(b)$ *of the quotient and subtract.*

(g.) *If there be no remainder, the root is completely extracted, but if more terms remain, proceed again according to* (c), (d), (e), (f), *remembering that both terms of the quotient are now to be dealt with.*

*This process, beginning with rule* (c), *must be continued till the entire root is found.*

### Examples XIII.

(1.) Find the cube roots of the following :—

$$\begin{array}{ll}
a^3+3a^2+3a+1 & x^3-3x^2+3x-1 \\
x^3-6x^2+12x-8 & a^6-9a^4+27a^2-27 \\
64x^3+48x^2+12x+1 & 8a^3+36a^2x+54ax^2+27a^3 \\
1-15ab+75a^2b^2-125a^3b^3 & -8x^6-48x^5-96x^4-64x^3.
\end{array}$$

(2.) Find the cube root of $x^6-6x^5+21x^4-44x^3+63x^2-54x+27$

*Working.*

$$\begin{array}{r|l}
 & x^6-6x^5+21x^4-44x^3+63x^2-54x+27(x^2-2x+3 \\
 & x^6 \\ \hline
3x^4-6x^3+4x^2) & -6x^5+21x^4-44x^3 \\
 & -6x^5+12x^4-\ \ 8x^3 \\ \hline
3(x^2-2x)^2=3x^4-12x^3+12x^2 & 9x^4-36x^3+63x^2-54x+27 \\
+3(x^2-2x).3+3^2=3x^4 & \\
-12x^3+21x^2-18x+9 & 9x^4-36x^3+63x^2-54x+27 \\ \hline
\end{array}$$

The learner should notice carefully the way in which the second divisor is formed : the quantity $(x^2-2x)$ is regarded as a single quantity at first, and is placed within brackets; the divisor is then formed by rules (c), (d), (e), and the whole quantity is then taken out of brackets and multiplied.

(3.) Find the cube roots of the following quantities :—

Of $x^6+9x^5+6x^4-99x^3-42x^2+441x-343$;

Of $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2s^7$
$+9xy^8+y^9$. ;

Of $27a^3-54a^2b-27a^2c+36ab^2+36abc+9ac^2-8b^3-12b^2c$
$-6bc^2-c^3$;

Of $a^9-3a^8+6a^7-10a^6+12a^5-12a^4+10a^3-6a^2+3a-1$;

Of $a^9+9a^8b^2+27a^7b^4+3a^7b+27a^6b^6+27a^6b^3+81a^5b^5+3a^5b^2$
$+81a^4b^7+27a^4b^4+81a^3b^6+a^3b^3+81a^2b^8+9a^2b^5+27ab^7$
$+27b^9$.

Of $\frac{a^3}{8}-9\frac{a^2}{9}+\frac{27a}{2}-27$

Of $\frac{8}{27}x^3-2x^2y+\frac{9}{2}xy^2-\frac{27}{8}y^3$

Of $64a^6x^3-12a^5x^4+\frac{3}{4}a^4x^5-\frac{a^3x^6}{64}$

Of $a^{6n}-3a^{5n}+6a^{4n}-7a^{3n}+6a^{2n}-3a^n+1$

Of $x^3+3x^2y-6x^2z+3xy^2-12xyz+12xz^2+y^3-6y^2z+12yz^2-8z^3$.

# CHAPTER VII.

## THE GREATEST COMMON MEASURE AND THE LEAST COMMON MULTIPLE.

**45.** *Def.* ANY algebraical quantity that divides another without remainder is called a measure of it.

Thus $a$ is a measure of $3a^2$
$7ab$ is a measure of $14abc$
and $(a+b)$ of $a^2+2ab+b^2$.

Any algebraical quantity that divides *two* others without remainder is called a *common measure* of these two.

Thus $ab$ is a common measure of $3a^2b$ and $6ab^2$.

The *greatest* algebraical quantity that divides two others without remainder is called the *greatest common measure* of these two.

Thus $3ab$ is the greatest common measure of $3a^2b$ and $6ab^2$.

Where the quantity consists of a single term, all its measures may be found by simple inspection.

Thus the measures of $12a^3b^2c$ are—

$2, 3, 4, 6, 12$; $a, 2a, 3a, 4a, 6a, 12a$; $ab, 2ab, 3ab, 4ab, 6ab, 12ab$; $abc, 2abc, 3abc, 4abc, 6abc, 12abc$; $a^2, 2a^2, 3a^2, 4a^2\ 6a^2, 12a^2$; $a^2b, 2a^2b, 3a^2b, 4a^2b, 6a^2b, 12a^2b$; &c.

If the learner writes down all the measures, he will find that the whole number of them is 143.

Write down all the measures of the following quantities :—

$8a^2$, $69ab$, $15a^2x^2y^2$; $ab^2c^4$, $30a^2$, $p^2q^2r^2$.

Find the G. C. M. of $4a^2x^2$ and $6abx$.

It is evident from inspection that $2ax$ is the *greatest* quantity that will divide both of the quantities without remainder,

$\therefore$ G. C. M. of $4a^2x^2$ and $6abx=2ax$.

### Examples XIV.

(1.) Find the G. C. M. of the following quantities :—

$7ab^2x^2$ and $14a^2x$; $a^2bc^2$ and $ab^2c$;
$-4a^3x^3$ and $10b^3y^3$; $12xyz$ and $-12x^2y^2z^2$;
$-48a^3b^2c$ and $-36ab^2c^3$; $10a^6b^5c^4x^3y^2z$ and $80ab^2c^3x^4y^5z^6$.

(2.) Find the G. C. M. of the three quantities—

$$4a^2bc,\ 6ab^2c,\ 5abc^2$$

The G. C. M. of $4a^2bc$ and $6ab^2c = 2abc$.
And the G. C. M. of $2abc$ and $5abc^2 = abc$.
And G. C. M. of $4a^2bc$, $6ab^2c$ and $5abc^2 = abc$.

Hence, to find the G. C. M. of three algebraical expressions, find the G. C. M. of the first two, which call $M$, then find the G. C. M. of $M$ and the third quantity; this will be the G. C. M. of the three expressions.

This rule may be extended to find the G. C. M. of four or any number of quantities.

(A proof of the rule will be given hereafter.)

Find the G. C. M. of the following quantities :—

$3x^2, 6x^3, 2x$; $4x^3y, 6xyz, 12xyz$; $7a^2b^2c^2, 14a^2b^2d^2, 21a^2c^2d^2, 21b^2c^2d^2$
$15p^3q^2r^2,\ 35p^2q^3r^2,\ 20p^2q^2r^3$;
$-9abcde,\ -18def,\ 12adf$; $-x^2y^2,\ x^2z^2,\ -y^2z^2$;
$4x^{\overline{m+2}}.y^{\overline{m+1}},\ 12x^{\overline{m+1}}.y^{\overline{m+2}}$; $8x^my^{2m}z^{3m},\ 12x^{2m}y^mz^{3m},\ 4x^{3m}y^{3m}z^{3m}$.

(3.) Find the G. C. M. of $a^2x^2$ and $a^2x+ax^2$.

Now $a^2x+ax^2=ax(a+x)$
And the G. C. M. of $a^2x^2$ and $ax(a+x)=ax$.

Similarly may be found the G. C. M. of the following :—

$3ax$ and $(6a^2-6ax)$; $(4a^2b-4ab^2)$ and $(2a^3-2a^2b)$;
$8a^2b^2c^2$ and $(4a^2bc-4ab^2c+4abc^2)$;
$ax(-x)$ and $a^2x^2(a^2+x^2)$; $6a^2b^2$, $12ab(a-b)$, $18ab(a+b)$;
$(a^2x-x^3)^2$, $(ax^2+x^3)$, $2ax^2$.

**46.** When the quantities both contain more than one term, it will be necessary to proceed differently.

The Rule in this case may be thus stated :—

*Arrange both the quantities according to the descending powers of some letter which is common to both; make the quantity of highest dimensions the dividend, and the other the divisor; work out the division as far as possible: then make the remainder a new divisor, and the former divisor a new dividend; work out the division again as far as possible, continue the former process with the remainder and divisor until there is no remainder. The last divisor will be the greatest common measure required.*

If $A$ and $B$ represent the two quantities whose G. C. M. is required, the dimensions of $A$ being equal to or greater than $B$, the process may be thus written :—

$$\begin{array}{l} B)\,A\,(p \\ \quad\underline{pB} \\ \qquad C)\,B\,(q \\ \qquad\quad \underline{qC} \\ \qquad\qquad D)\,C\,(r \\ \qquad\qquad\quad \underline{rD} \\ \qquad\qquad\quad \underline{\;\cdot\,\cdot\;} \end{array}$$

Here $A$ contains $B$, $p$ times with a remainder $C$.
$B$ ,, $C$, $q$ ,, ,, ,, $D$.
$C$ ,, $D$, $r$ times exactly.

It is now required to prove that $D$ is the G. C. M. of the two quantities $A$ and $B$.

The proof of this rule depends upon the truth of these two propositions.

1°. *If one quantity measure another, it will also measure any multiple of that other.*

Let $X$ measure $Y$, $m$ times

i.e. $Y = m.X$

then $nY = mn.X$.

or, $X$ measures $nY$, $mn$ times.

*Example.* $5x$ measures $10x^2$.
Also, $5x$ measures $20x^2$, $30x^2$, $40x^3$, which are multiples of $10x^2$.

2°. *If one quantity measure two others, it will also measure the sum or the difference of these two.*

Let $X$ measure $Y$, $a$ times and $Z$, $b$ times.

So that $Y = aX$

$Z = bX$

Then $Y+Z = aX+bX$

or, $Y+Z = (a+b)X$

and $Y-Z = aX-bX$

or, $Y-Z = (a-b)X$

*Example.* $3x^2$ measures $15x^3$ and $9x^2$

$\therefore$ $9x^2+15x^3=3x^2(3+5x)$

or, $3x^2$ measures $9x^2+15x^3$

Also, $15x^3-9x^3=3x^2(5x-3)$

or, $3x^2$ measures $15x^3-9x^2$.

Now, by referring to the division, we see that—

$$A=pB+C$$
$$B=qC+D$$
$$C=rD.$$

That is $B=qrD+D$

or, $B=(qr+1)D$; $\therefore$ $D$ measures $B$

$A=p.(qr+1)D+rD$

or, $A=(pqr+p+r).D$; $\therefore$ $D$ measures $A$.

$D$ is therefore *a common measure of* $A$ and $B$.

We must now show that $D$ is the *greatest common measure* of $A$ and $B$.

For if $D$ be not the G. C. M. of $A$ and $B$
let $E$ be the G. C. M.

Since $E$ measures $A$ and $B$
$\therefore$ $E$ measures $A$ and $pB$ and $A-pB$ by Prop. (2).
That is $E$ measures $C$; for $A-pB=C$
$\therefore$ $E$ measures $qC$ and $B-qC$
or, $E$ measures $D$; for $B-qC=D$
$\therefore$ $E$ the G. C. M. of $A$ and $B$ cannot be greater than $D$.

## Examples XV.

(1.) Find the G. C. M. of $x^2-7x+12$ and $x^2-8x+16$.

*Working.*

$$\begin{array}{r}
x^2-7x+12)\,x^2-8x+16\,(1\\
\underline{x^2-7x+12}\\
-x+4)\,x^2-7x+12\,(-x+3\\
\underline{x^2-4x\phantom{+12}}\\
-3x+12\\
\underline{-3x+12}\\
\cdot\quad\cdot
\end{array}$$

$\therefore$ G. C. M. $=-x+4$

or, $x-4$, if we multiply $(-x+4)$ by $(-1)$, which is always allowable.

(2.) Find the G. C. M. of the following quantities :—

$x^2+6x+9$ and $x^2+5x+6$
$x^2-15+56$ and $x^2-16x+64$
$x^4+x^2-12$ and $x^4-9$
$x^2+3x+2$ and $x^3+3x^2+3x+1$
$x^3-5x^2+5x-4$ and $x^2-5x+4$
$6x^2+7x+2$ and $12x^2+16x+5$.

(3.) Find the G. C. M. of $3x^2+4x+1$ and $3x^3-2x^2-3x+2$.

*Working.*

$$\begin{array}{r}
3x^2+4x+1)\,3x^3-2x^2-3x+2\,(x-2\\
\underline{3x^3+4x^2+x\phantom{+2}}\\
-6x^2-4x+2\\
\underline{-6x^2-8x-2}\\
4)\,4x+4\\
x+1)\,3x^2+4x+1\,(3x+1\\
\underline{3x^2+3x\phantom{+1}}\\
x+1\\
\underline{x+1}\\
\cdot\quad\cdot
\end{array}$$

$\therefore$ G. C. M. $=x+1$.

*Note.*—In this example, the remainder ($4x+4$) is divided by 4; it then found that ($x+1$) divides into ($3x^2+4x+1$), and the result o tained by dividing by ($x+1$) will be correct; for it is evident that th G. C. M. of ($x+1$) and ($3x^2+4x+1$) is the same as the G. C. M. of $4x+$ and $3x^2+4x+1$); since the factor 4, which is omitted, is no part of th G. C. M. And in all cases, the remainder should be divided by a number or simple algebraical quantity which will divide all its terms.

(4.) Find the G. C. M. of the following:—

$a^2-b^2$ and $(a+b)$; $x^2-5x+6$ and $x^2-3x+2$

$x^2+6x-7$ and $x^2-8x+7$; $(x-1)^2$ and $(x^2-1)^2$

$x^2-x-12$ and $x^2-8x+16$; $x^2-100$ and $x^2-x-90$.

$x^3-11x+20$ and $x^3-5x^2+9x-5$

$7x^3-3x^2+28x-12$ and $14x^2+x-3$

$2x^4+5x^2-12$ and $x^4+x^2-12$

$x^3-22x-15$ and $5x^3+24x^2+10x-3$

$24x^3+46x^2+29x+6$ and $24x^3-22x^2-11x+6$

$6x^4+10x^3-12x^2-17x+5$ and $12x^4+17x^3-5x^2+3x+5$

$x^2-7x+10$ and $x^4-49x^2+140x-100$

$3a^2-10ab+8b^2$ and $3a^2+4ab-4b^2$

$4x^4+14x^3y+6x^2y^2-21xy^3-18y^4$ and $2x^3+14x^2y-3xy^2-21y^3$.

(5.) Find the G. C. M. of $12x^3-13x^2y-11xy^2+2x^3$ and

***Working.*** $6x^4+25x^3y+14x^2y^2-3xy^3-2y^4$

$$
\begin{array}{r}
2 \\
\hline
12x^3-13x^2y-11xy^2+2y^3)\;12x^4+50x^3y+28x^2y^2-6xy^3-4y^4\;(x \\
12x^4-13x^3y-11x^2y^2+2xy^3 \\
\hline
63x^3y+39x^2y^2-8xy^3-4y^4 \\
4 \\
\hline
252x^3y+156x^2y^2-32xy^3-16y^4\;(21y \\
252x^3y-273x^2y^2-231xy^3+42y^4 \\
\hline
y^2)\;429x^2y^2+199xy^3-58y^4 \\
\hline
429x^2+199xy-58y^2
\end{array}
$$

$$
\begin{array}{r}
429x^2+199xy-58y^2)\;1716x^3-1859x^2y-1573xy^2+286y^3\;(4x \\
1716x^3+796x^2y-232xy^2 \\
\hline
-2655x^2y-1341xy^2+286y^3 \\
-143 \\
\hline
379665x^2y+191763xy^2-40898y^3\;(885y \\
379665x^2y+176115xy^2-51330y^3 \\
\hline
5216y^2)\;15648xy^2+10432y^3 \\
\hline
3x+2y)\;429x^2+199xy-58y^2\;(143x \\
429x^2+286xy \\
\hline
-87xy-58y^2 \\
-87xy-58y^2 \\
\hline
\cdot \quad \cdot \\
\hline
\end{array}
$$

G. C. M. $=3x+2y$

This example should be carefully studied, and the following points observed :—

1°. One of the quantities is multiplied by 2, in order that its first term $12x^4$ may exactly contain the first term of the divisor $12x^3$; similarly, the first remainder is multiplied by 4, in order to make its first term divisible by that of the divisor.

2°. The second remainder is divided by $y^2$, before it is used as a divisor.

Similar observations may be made on the second part of the process; and a like method must be pursued whenever the remainder or divisors require it.

(6.) Find the G. C. M. of the following quantities :—

$2x^4+x^3+5x^2+2x-18$ and $4x^3+7x^2-8x-14$
$21x^2-47x-24$ and $6x^3-16x^2+9x-24$
$4a^3-9a^2b-16ab^2+21b^3$ and $3a^4-6a^3b-11a^2b^2+4ab^3+6b^4$
$110a^2-221ab+110b^2$ and $110a^2+21ab-110b^2$
$24a^4-22a^3b-17a^2b^2+23ab^3-6b^4$ and
$24a^4-14a^3b-23a^2b^2-22ab^3+6b^4$
$48a^4+68a^3b-84a^2b^2-41ab^3+30b^4$ and
$105a^4-34a^3b-160a^2b^2+208ab^3-96b^4$.

(7.) Find the G. C. M. of $6a^2x-6x^3$ and $3a^2x^2+18ax^3+15x^4$.

*Working.*

$$6x)\underline{6a^2x-6x^3} \qquad 3x^2)\underline{3a^2x^2+18ax^3+15x^4}$$
$$\underline{a^2-x^2} \qquad \underline{a^2+6ax+5x^2}$$

$$
\begin{array}{l}
a^2-x^2)a^2+6ax+5x^2(1 \\
\qquad\quad \underline{a^2 \qquad\quad -x^2} \\
\qquad\quad 6x)\underline{6ax+6x^2} \\
\qquad\qquad\quad a+x)a^2-x^2(a-x \\
\qquad\qquad\qquad\quad \underline{a^2+ax} \\
\qquad\qquad\qquad\quad -ax-x^2 \\
\qquad\qquad\qquad\quad \underline{-ax-x^2} \\
\qquad\qquad\qquad\qquad \cdot \quad \cdot
\end{array}
$$

$a+x$ is therefore the G. C. M. of $a^2-x^2$ and $a^2+6ax+5x^2$, but the original quantities were divided by $6x$ and $3x^2$ respectively, before the operation began, and these contain a common factor $3x$.

Hence the G. C. M. of

$$6a^2x-6x^3 \text{ and } 3a^2x^2+18ax^3+15x^4=3x(a+x) \text{ or } 3ax+3x^2.$$

*Note.*—Any factors which can be discovered by inspection should be removed from both the quantities at first, and if any of such factors be common to both the quantities, they must be introduced into the G. C. M. as in the last example.

(8.) Find the G. C. M. of—

$4x^2+32x+28$ and $6x^2-24x-30$
$7x^4y+7x^2y^3$ *and* $5x^5y^3-5xy^6$

$2x^3-8x^2y+16xy^2$ and $8x^2-4xy-24y^2$
$3x^2-3x-36$ and $7x^3-56x^2+112x$
$5x^3-55x+100$ and $10x^4-50x^3+90x^2-50x$
$14x^3y^3-6x^2y^3+56xy^3-24y^3$ and $42x^2y+3xy-9y$
$3x^3-24x^2-36x+432$ and $6x^4-32x^3-24x^2$.

**47.** To find the G. C. M. of three algebraical quantities $X$, $Y$, $Z$.

### RULE.

*First find the* G. C. M. *of $X$ and $Y$; call this $M$: then find the* G. C. M. *of $M$ and $Z$; this shall be the* G. C. M. *of $X$, $Y$, $Z$.*

To prove this, consider—

1°. That every common measure of $X$, $Y$ is a measure of $M$;* and, therefore, every common measure of $X$, $Y$, $Z$ is a common measure of $M$ and $Z$.

2°. That every measure of $M$ is a common measure of $X$ and $Y$ (page 42), and, therefore, every common measure of $M$ and $Z$ is a common measure of $X$, $Y$, $Z$.

Hence all the common measures of $M$ and $Z$ are the same as the common measures of $X$, $Y$, $Z$, and .·. the G. C. M. of $M$ and $Z$ must be the G. C. M. of $X$, $Y$, $Z$.

**48.** To find the G. C. M. of four quantities $X$, $Y$, $Z$, $W$.

### RULE.

*Find the* G. C. M. *of $X$, $Y$, $Z$ by the last rule; call this $N$: then find the* G. C. M. *of $N$ and $Z$: this shall be the* G. C. M. *of the four quantities.*

The proof of this rule is similar to the last; it should be written out by the pupil.

A similar method is to be used when the G. C. M. of five or six, or any number of quantities, is required.

### Examples XVI.

(1.) Find the G. C. M. of—

$4x^2$, $10xy^2$, $20x^2y^2$
$-a^3b^2c$, $3a^2b^3c$, $-10a^2bc$
$14a^2$, $10b^2$, $16c^2$, $2d^2$
$24xyz$, $120x^2y^2z^2$, $48x^3y^3z^3$
$7abc^2d^2$, $12a^2b^2cd$, $24a^2bcd^2$, $14ab^2c^2d$.
$448p^3q^4r^5$, $560p^5q^4r^3$, $480p^4q^4r^4$.

---

* For if it were not, $X$ and $Y$ would have a common factor, $P$ suppose, not contained in $M$, and, therefore, $X$ and $Y$ would be divisible by the product of $P$ and $M$, or their G. C. M. is greater than $M$, which is impossible.

(2.) Find the G. C. M. of—

$4x^2y$, $10xy^2$ and $4x^3y-xy^3$
$(x+y)^2$, $4x+4y$, $6x^2+12xy+6y^2$
$(x^2-xy)^2$, $x^2+xy)^2$, $x^2.(x^2-y^2)^2$
$15xyz$, $3(x^2+xz)^3$, $24(x^2-xz)^3$
$a^2+2ab+b^2-c^2$, $a^2+2ac+c^2-b^2$, $b^2+2bc+c^2-a^2$
$(x^4-y^4)$, $(x^2+y^2)^2$, $4x^3+4xy^2$, $(x^2y^2+y^4)^3$
$x^2+6x+9$, $x^2+5x+6$ and $x^2+2x-3$
$12x^2+16x+5$, $8x^3+4x^2+6x+3$ and $6x^2+7x+2$
$x^4-10x^3-4x+40$, $x^2-100$ and $x^2-x-90$
$21a^2+ab-10b^2$, $3a^2+4ab-4b^2$ and $3a^2-10ab+8b^2$
$2x^4+x^2-28$, $x^4+x^2-12$ and $2x^4+5x^2-12$
$12x^3+17x^2+3x-2$, $12x^3+23x^2+13x+2$ and $3x^3-3x^2-3x+3$
$x^4-13x^2+36$, $x^4+x^3-11x^2-9x+18$, $x^4-10x^2+9$ and
$x^4-x^3-13x^2+x+12$.

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## THE LEAST COMMON MULTIPLE.

**49.** A "Common Multiple" of two algebraical quantities is one that contains them both a certain number of times exactly.

The "Least Common Multiple" of two quantities is the least quantity that contains them both a certain number of times exactly.

Thus 12 is the L. C. M. of 6 and 4,
and $15a^2b^2$ is the L. C. M. of $3ab^2$ and $5a^2b$.

To find the L. C. M. of two quantities, observe the following

### RULE.

*Multiply the two quantities together, and divide this product by their* G.C.M.

This rule may be proved as follows:—

Let $XY$ be the two quantities
$M$ their G. C. M.
Also, let $X=MP$
$Y=MQ$.

Then $PQ$ can have no common factor, for if they had, $M$ could not be the G. C. M. of $XY$.

Now $MPQ$ contains $MP$ or $X$ and $MQ$ or $Y$; therefore, $MPQ$ is *a common multiple* of $XY$. Also, it is the least common multiple of $XY$. For the L. C. M. must contain all the factors of $M$ and $P$ together, also it must contain all the factors of $M$ and $Q$ together; but since $P$ and $Q$ have no common factor, it must therefore contain all the factors of $MPQ$ together, and $MPQ$ is the least quantity that contains all these.

Hence $MPQ$ is the L. C. M. of $X$ and $Y$.

$$\text{But}\qquad MPQ=\frac{MP\times MQ}{M}$$

$$=\frac{X.Y}{M}\ \text{which is the L. C. M.}$$

## Examples XVII.

Find the L. C. M. of $15a^2bc$ and $12ab$.

*Working.* G. C. M. of $15a^2bc$ and $12ab$

$$\begin{aligned} &=3ab\\ \therefore \text{L. C. M. } &=\frac{15a^2bc\times 12ab}{3ab}\\ &=15a^2bc\times 4\\ &=60a^2bc. \end{aligned}$$

Also of $3(x^2-a^2)$ and $2x^2+2ax$.

Here the G. C. M. is found to be $x+a$

$$\begin{aligned} \therefore \text{L. C. M. } &=\frac{3(x^2-a^2)\times 2x(x+a)}{x+a}\\ &=3(x^2-a^2)\times 2x\\ &=6x(x^2-a^2)=6x^3-6a^2x. \end{aligned}$$

*Note.*—In the fractional expression for the L. C. M. the denominator should always be cancelled against some of the factors in the numerator, before the multiplication is effected. Also, great care should be taken to work out all the examples in a form similar to the two just given.

(2.) Find the L. C. M. of $a^3-x^3$ and $4a^4+4a^2x^2+4x^4$.

Removing the factor 4 from the latter, we proceed to find the G. C. M.

$$\begin{array}{l} a^3-x^3)\,a^4+a^2x^2+x^4\,(a\\ \qquad\quad\ \ \underline{a^4-ax^3\qquad\qquad}\\ \qquad\quad x^2)\,\underline{a^2x^2+ax^3+x^4}\\ \qquad\qquad\quad a^2+ax+x^2)\,a^3-x^3\qquad(a-x\\ \qquad\qquad\qquad\qquad\qquad\ \ \underline{a^3+a^2x+ax^2\quad}\\ \qquad\qquad\qquad\qquad\qquad\quad -a^2x-ax^2-x^3\\ \qquad\qquad\qquad\qquad\qquad\quad \underline{-a^2x-ax^2-x^3}\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \cdot\quad\ \cdot \end{array}$$

$\therefore$ G. C. M. $a^2+ax+x^2$.

And by the rule—

$$\text{L. C. M.} = \frac{(a^3-x^3)(4a^4+4a^2x^2+4x^4)}{(a^2+ax+x^2)}$$
$$= (a-x)(4a^4+4a^2x^2+4x^4).$$

$(a-x)$ being obtained by cancelling $(a^2+ax+x^2)$ against $a^3-x^3$. By multiplication the result is known to be—

$$\text{L. C. M.} = 4a^5-4a^4x+4a^3x^2-4a^2x^3+4ax^4-4x^5.$$

All the examples should be worked out in this form.

For examples of L. C. M. the reader is referred to those in the G. C. M., each of which is well adapted for an exercise in finding the least common multiple.

# CHAPTER VIII.

## ABRIDGED MULTIPLICATION AND DIVISION.

**50.** By actual multiplication we find that—

$$(a+b)\times(a-b)=a^2-b^2.$$

Here $(a+b)$ represents the *sum* of the two quantities $a$ and $b$.
And $(a-b)$ represents the *difference* of $a$ and $b$.
And $(a^2-b^2)$ represents the *difference* of the *squares* of $a$ and $b$.

Hence the above formula, put into words, gives the following Rule—

*The product of the sum and the difference of any two quantities equals the difference of their squares.*

Thus
$$(3a+x).(3a-x)=(3a)^2-x^2$$
$$=9a^2-x^2$$

$$(5xy+3ab).(5xy-3ab)=(5xy)^2-(3ab)^2$$
$$=25x^2y^2-9a^2b^2.$$

### Examples XVIII.

(1.) Write down the products of the following, without the work of multiplication :—

$4a+1$ and $4a-1$; $6a^2-1$ and $6a^2+1$
$7ab+4xy$ and $7ab-4xy$; $x^3+y^2$ and $x^3-y^2$

$a^3+b^3$ and $a^3-b^3$; $-a^4+b^4$ and $a^4+b^4$
$-a^2-b^2$ and $-a^2+b^2$; $3a^2x^2-4b^2y^2$ and $3a^2x^2+4b^2y^2$
$5x^2y+2xy^2$ and $2xy^2-5x^2y$.

(2.) Find the product of $a+b+c$ and $a+b-c$.

*Working.*—Regarding $\overline{a+b}$ in the first instance as a single quantity, and placing a vinculum over it, we have

$$\begin{aligned}(a+b+c)\,(a+b-c) &= (\overline{a+b}+c)\,(\overline{a+b}-c).\\ &= (\overline{a+b}|^2-c^2\\ &= a^2+2ab+b^2-c^2.\end{aligned}$$

Again, the product $a+b+c+d$ and $a+b-c-d$ may be thus found:—

$$(a+b+c+d)\,(a+b-c-d) = (\overline{a+b}+\overline{c+d})\,(\overline{a+b}-\overline{c+d})$$
$$=(\overline{a+b})^2-(\overline{c+d})^2=a^2+2ab+b^2-c^2-2cd-d^2.$$

*Note.* Any two factors which contain the same letters and the same coefficients may be arranged as a sum and difference, and their product found by the above Rule.

Thus the two factors $3a-4b-4c+5d$ and $3a+4b-4c-5d$ are equal to $(\overline{3a-4c}-\overline{4b-5d})$ and $(\overline{3a-4c}+\overline{4b-5d})$ respectively.

Find the product of—

$a+b+c$ and $a+b-c$
$a+b+c$ and $a-b-c$
$a+b+c$ and $c-a-b$
$a+c-b$ and $b-c-a$
$a+b+c$ and $b+c-a$
$a+b+c$ and $b-a-c$
$a+b-c$ and $a-b-c$
$b+c-a$ and $a-b-c$

Also of $a+b+c+d$ and $a-b-c-d$
$b+c-d-a$ and $a-b+c-d$
$1-x+x^2-x^3+x^4$ and $1+x+x^2+x^3+x^4$
$1-2x+3x^2+4x^3$ and $1-2x-3x^2+4x^3$.

Find the continued product of—

$x+y+z$, $x+y-z$, and $x^2+y^2-z^2+2xy$
$1+x+x^2$, $1-x+x^2$, and $1-x^2-y^4$.

**51.** By actual multiplication of the factors $(x+a)$ and $(x+b)$ as below, we find their product to be—

$$x^2+(a+b)\,x+ab \qquad \begin{array}{l} x+a\\ x+b\\ \hline x^2+ax\\ \quad\;+bx+ab\\ \hline x^2+(a+b)\,x+ab \end{array}$$

Whence it appears that the coefficient of $x$ in the product is the sum of the last terms of each factor, and the last term of the product is the product of the two last terms of each factor.

$$\begin{aligned}
\text{Thus}\quad (x+7)(x+9) &= x^2+(7+9)x+7\times 9\\
&= x^2+16x+63\\
\text{Again,}\quad (x+2)(x-5) &= x^2+(2-5)x+2\times(-5)\\
&= x^2-3x-10\\
(x-4)(x-7) &= x^2+(-4-7)x+(-4)\times(-7)\\
&= x^2-11x+28.
\end{aligned}$$

Similarly, may the products of the following be found:

$x+13$ and $x-1$, $x+8$ and $x-7$, $x-14$ and $x-13$
$x+4$ and $x-4$, $x-9$ and $x-10$, $x+10$ and $x+1$
$a-3$ and $a-2$, $a+7$ and $a-8$, $a-3$ and $a+4$
$a-4b$ and $a+7b$, $a-b$ and $a-2b$, $a+7b$ and $a+10b$
$x+y$ and $x+4y$, $x-y$ and $x-7y$, $x^2+y^2$ and $x^2+4y^2$
$a^2-3b^2$ and $a^2-4b^2$, $a^2+10b^2$ and $a^2-11b^2$
$x^2-4ab$ and $y^2-7ab$, $x^3-10y^3$ and $x^3+2y^3$.

Also, the continued products of the following factors:—

$x+4y$, $x-4y$, and $x^2+y^2$

$$\begin{aligned}
\text{For,}\quad (x+4y)(x-4y) &= x^2-16y^2 \text{ by Rule (p. 49)}\\
\text{And,}\quad (x^2-16y^2)(x^2+y^2) &= x^4+(1-16)x^2y^2-16y^4\\
&= x^4-15x^2y^2-16y^4.
\end{aligned}$$

$a+3b$, $a-3b$, and $a^2-3b^2$
$4a+x$, $x-4a$, and $a^2+2x^2$
$7a-b$, $7a^2+b^2$, and $7a+b$
$x^2-2y^2$, $x^4-4y^4$ and $2y^2+x^2$
$(x+y)^2$ and $(x-y)^2$, $(a+b)^3$ and $(a-b)^3$
$(a+b+c$, $a+b-c$ and $a^2-2ab-b^2-c^2$.

**52.** It has been shown (p. 29) that $a^m-b^m$ is always divisible by $a-b$, and the form of the quotient was there given. Hence the quotient of any quantity of the form $a^m-b^m \div a-b$ may at once be written down without working out the division.

$$\begin{aligned}
\text{Thus,}\quad & a^4-b^4 \div a-b = a^3+a^2b+ab^2+b^3\\
\text{Also, since}\quad & a^4-b^4 = (a-b)(a^3+a^2b+ab^2+b^3),
\end{aligned}$$

it is always possible to write down the factors of such a quantity.

## Examples XIX.

Resolve into their factors the following quantities :—

$a^3-1$, $x^5-1$, $4x^2-9y^2$, $32x^5-243$, $x^7-y^{14}$, $8a^3x^3-27b^6$, $27a^3b^6c^9-1$, $64p^3-q^3$, $81a^4x^4-256b^8$.

It was also shown in the same Article that

$a^m-b^m$ is divisible by $a+b$, when $m$ is even,
$a^m+b^m$ is divisible by $a+b$, when $m$ is odd.

$\therefore$ $a^m-b^m$ is divisible by both $(a-b)$ and $(a+b)$ in the former case.

Thus

$$\begin{aligned} a^6-b^6 &= (a^3-b^3)\,(a^3+b^3) \\ &= (a-b)\,(a^2+ab+b^2).(a+b)\,(a^2-ab+b^2) \\ 81a^4-b^8 &= (3a)^4-(b^2)^4 \\ &= \{\,(3a)^2-(b^2)^2\}.\{\,(3a)^2+(b^2)^2\} \\ &= (3a-b^2)\,(3a+b^2)\,(9a^2+b^4). \end{aligned}$$

Again,

$$\begin{aligned} (a^4-2a^2b^2+b^4 &= (a^2-b^2)\,(a^2-b^2) \\ &= (a-b)\,(a+b)\,(a-b)\,(a+b). \end{aligned}$$

Resolve into all their factors the following quantities :—

$x^5-a^5$, $64a^6-x^6$, $a^8x^8-b^8y^8$, $256x^4-16y^4$, $729a^6b^6c^6+x^3y^3z^3$, $64+a^6$, $a^{10}+b^{10}$, $1024a^{10}-1$, $p^3q^6r^9-27$, $216x^6-27$, $x^8-y^8$, $x^9+y^9$, $4a^5-4a^2x^3$, $16x^3+2$, $a^8-a^4x^4$, $192+3a^6$, $a^3x^3-x^6$, $58b^2-8a^2$, $p^9+64p^3q^6$, $4x^7+128a^5x^2$.

### 53. To resolve into factors a trinomial, as—

$$x^2+5x+4$$

In Article 51 it was shown that $x^2+(a+b)\,x+ab$ is the product of two factors $(x+a)$, $(x+b)$. Now, since $x^2+5x+4$ is of the some form as $x^2+(a+b)\,x+ab$, let us assume the two factors of it to be $(x+a)$ and $(x+b)$. Then we must have—

$$\begin{aligned} a+b&=5 \\ ab&=4 \end{aligned}$$

And we must now find by inspection such values of $a$ and $b$ as shall fulfil these two conditions; we thus find—

$$\begin{aligned} a&=4 \\ b&=1 \end{aligned}$$

Similarly, to find the factors of

$$x^2-x-12$$

we must have

$$\begin{aligned} a+b &= -1 \\ ab &= -12 \end{aligned}$$

Therefore, $a$ and $b$ must be of *opposite* signs, and the negative quantity is the greater.

Hence $$\left.\begin{aligned} a &= -4 \\ b &= \phantom{-}3 \end{aligned}\right\}$$

Or, $$x^2-x-12 = (x-4)(x+3).$$

### Examples XX.

Resolve into factors the following quantities :—

$x^2+7x+10$

$x^2+3x-28$

$x^2-11x+10$

$x^2+11x+30$

$x^2+3xy+2y^2$

$p^2+pq-20q^2$

$x^3-8x^2+7x$

$x^5-5x^4y+6y^2$

$4a^3b-28a^2b^2-72ab^3$

$b^2x^2y^2-7abx^3y-18a^2x^4$

$x^4y^2+3x^3y^3-28x^2y^4$

$x^2+14x+24$

$x^2-3x-28$

$x^2-4x-21$

$x^2+x-30$

$x^2-7xy+12y^2$

$p^2-7pq+12q^2$

$x^4+21x^3+110x^2$

$x^2+x-90$

$x^2-11x-12$

$x^2+4x-21$

$x^2-x-30$

$x^2+7xy+10y^2$

$p^2-7pq+6q^2$

$y^3-10y^2+21y$

$a^2b^2-ab^3-420b^4$

$x^4+16x^2y+63y^2$

$a^6-2a^3b^2-63b^4$

$x^2y^2z^2+7xyz-30$

# CHAPTER IX.

## FRACTIONS.

**54.** IN Arithmetic, the Denominator of a Vulgar Fraction shows into how many parts the whole is divided, and the Numerator shows how many of those parts are taken. Thus, the fraction $\frac{3}{8}$ denotes that the unit is divided into 8 equal parts, and that 3 of those are taken.

Similarly, in Algebra, the fraction $\dfrac{a}{b}$ denotes that the unit is divided into $b$ equal parts, and that $a$ of these are taken; $\dfrac{a-x}{a+x}$, that the unit is divided into $\overline{a+x}$ equal parts, and that $\overline{a-x}$ of these are taken.

Also, in Algebra as in Arithmetic, any fraction may be supposed to represent the quotient arising from the division of the numerator by the denominator.

Thus $\frac{a}{b} = a \div b$

For $\frac{1}{b}$ means the $b$th part of unity,

And $\frac{a}{b}$ means $a$ of those parts.

Again, $a \div b$ means the $b$th part of $a$, which must be $a$ times as great as the $b$th part of unity.

Hence $\frac{a}{b} = a \div b$

**55.** To represent any integral quantity as a fraction with a given denominator.

RULE.

*Multiply the quantity by the given denominator.*

For $a = ab \div b = \frac{ab}{b}$

That is, to reduce $a$ to a fraction with denominator $b$, we must multiply the $a$ by $b$ for the numerator.

This rule is chiefly applied to such cases as the following:—

To find the sum of $a + \frac{3a^2b}{ab}$

Now, $a + \frac{3a^2b}{ab} = \frac{a.ab}{ab} + \frac{3a^2}{ab} = \frac{a^2b+3a^2b}{ab} = \frac{4a^2b}{ab}$

Again, $1 - \frac{a^2+b^2-c^2}{2ab} = \frac{2ab-(a^2+b^2-c^2)}{2ab} = \frac{c^2-a^2+2ab-b^2}{2ab}$

### Examples XXI.

Reduce to their simplest forms the following :—

$$a + \frac{b^2}{a},\ 1 - \frac{b}{a},\ \frac{a^2}{b^2} - 1,\ \frac{a^2+2ab}{b^2} + 1,\ a^2 - \frac{4b^2}{c^2},$$

$$\frac{x-y}{x+y} + 1,\ \frac{x^2+y^2}{x^2-y^2} - 1,\ \frac{x^2+y^2}{x+y} - x,\ \frac{a^2+b^2}{2a-2b} - a+b$$

*Working.* $(2a-2b)(-a+b) = -2a^2+4ab-2b^2$.

$$\therefore \frac{a^2+b^2}{2a-2b} - a+b = \frac{a^2+b^2-2a^2+4ab-2b^2}{2a-2b} = \frac{4ab-a^2-b^2}{2a-2b}$$

$$\frac{7a^2+8b^2}{7a+8b}+a-2b,\quad \frac{x^4+a^4}{a+x}-(x^3-3a^2x+3ax^3-a^3)$$

$$\frac{5x^3}{x^2-1}-(x+1),\quad \frac{1}{x+1}-1,\quad \frac{(a+b)^2+c^2}{a+b+c}-(b+c-a).$$

**56.** i. To prove that $\frac{a}{b}\times c=\frac{ac}{b}$

For $\frac{a}{b}$ represents the $b$th part of $a$, and $\frac{ac}{b}$ represents the $b$th part of $ac$; but $ac$ is $c$ times as great as $a$, and therefore $\frac{ac}{b}$ is $c$ times as great as $\frac{a}{b}$.

*Hence, to multiply a fraction by any integral quantity, multiply the numerator by that quantity, and retain the same denominator.*

ii. To prove that $\frac{a}{bc}\times c=\frac{a}{b}$

For, in the former fraction $\frac{a}{bc}$, $a$ is divided into $bc$ equal parts, and $\frac{a}{bc}$ is one of those parts; in the latter case, $a$ is divided into $b$ equal parts, and $\frac{a}{b}$ is one of those parts, and each of the parts in the latter case is $c$ times as great as in the former case, or

$\frac{a}{b}$ is $c$ times as great as $\frac{a}{bc}$.

*Hence, to multiply a fraction by any integral quantity, we may divide the denominator by that quantity and retain the same numerator.*

iii. To prove that $\frac{ac}{b}\div c=\frac{a}{b}$

Repeat the reasoning of (i.).

*Hence, to divide a fraction by any integral quantity, divide the numerator by that quantity, and retain the same denominator.*

iv. To prove that $\frac{a}{b} \div c = \frac{a}{bc}$

Repeat the reasoning of (ii.).

*Hence, to divide any fraction by any integral quantity, multiply the denominator by that quantity, and retain the same numerator.*

*Note.* Rules (ii.) and (iii.) are not always possible; but when they are so, these should be used in preference to (i.) and (iv.).

### Examples XXII.

Multiply $\frac{a^2x^2}{b^2c^2}$ by $bc$.

By (i.) $\frac{a^2x^2}{b^2c^2} \times bc = \frac{a^2b^2c^2x^2}{b^2c^2}$

By (ii.) $\frac{a^2x^2}{b^2c^2} \times bc = \frac{a^2x^2}{bc}$

Divide $\frac{a^2-x^2}{ab}$ by $a+x$.

By (iii.) $\frac{a^2-x^2}{ab} \div a+x = \frac{a-x}{ab}$

By (iv.) $\frac{a^2-x^2}{ab} \div a+x = \frac{a^2-x^2}{ab\,(a+x)}$

Here the advantage of (ii.) and (iii.) is seen in giving the answer in lower terms.

(1.) Multiply $\frac{abc}{xy}$ into $xy$, $\frac{xy}{abc}$ into $xy$, $\frac{a}{bcd}$ into $bd$,

$\frac{a^2+x^2}{ax}$ into $x$, $\frac{a^3-x^3}{a^2+x^2}$ into $4\,(a+x)$

(2.) Find the following quotients :—

$\frac{axy}{bc} \div ab$, $\frac{x+y}{x-y} \div (x+y)$, $\frac{x^3-y^3}{x^2+y^2} \div (x-y)$

$\frac{a^3+x^3}{a-x} \div (a^2-ax+x^2)$, $\frac{7abcxy}{(ab+cy)} \div (ab-cy)$

(3.) Find the following products and quotients :—

$\frac{5a^2-5b^2}{7ab} \times 14a^2$, $\frac{9abc}{7a+7b+7c} \times (a+b+c)$, $\frac{3x^2y^2}{8\,(x+y)} \times (4x+4y)$

$$\frac{(a-b)^3}{a^3b^3} \times (ab)^2, \quad \frac{(x+y+z)^2}{(x-y+z)^3} \times 3\,(x-y+z),$$

$$\frac{25x^2-4y^2}{10xy} \times 20x^2y^2, \quad \frac{7abc}{16a^4-9b^4} \times (4a^2-3b^2),$$

$$\frac{(x+y)^2}{x^2-y^2} \div (x-y), \quad \frac{abxy}{x^3-y^3} \div (x^3+y^3), \quad \frac{x^2\,(x^3+a^3)}{x^2+ax+a^2} \div (x-a)$$

**57.** If the numerator and denominator of a Fraction be both multiplied by the same quantity, the value of the Fraction is not changed.

Thus $$\frac{a}{b} = \frac{ac}{bc}$$

For, the effect of multiplying the numerator by $c$ is to *multiply* the whole fraction by $c$, and the effect of multiplying the denominator by $c$ is to *divide* the whole fraction by $c$; but when a quantity has been both multiplied and divided by the same number or quantity, its value is not changed.

$$\therefore \frac{a}{b} = \frac{ac}{bc}.$$

For the same reasons it appears, that if the numerator and denominator be both *divided* by the same quantity, the value of the Fraction is not changed, that is

$$\frac{ac}{bc} = \frac{a}{b}$$

**58.** To reduce a Fraction to its lowest terms.

RULE.

*Divide the numerator and denominator by their* G. C. M.

For, since numerator and denominator are both divided by the same quantity, the value of the fraction will not be changed; and since the divisor is the G. C. M., no common factors can remain in the numerator and denominator, or the fraction will be in its lowest terms.

Reduce $\frac{4abc}{10a^2b^2c^2}$ to its lowest terms.

Here G. C. M. $= 2abc$.

And $$\frac{4abc}{10a^2b^2c^2} = \frac{2abc \times 2.}{2abc \times 5abc} = \frac{2}{5abc}$$

Reduce $\dfrac{a^3b-ab^3}{a^4+2a^3b+a^2b^2}$ to its lowest terms.

Here G. C. M. $= a(a+b)$.

And $\dfrac{a^3b-ab^3}{a^4+2a^3b+a^2b^2} = \dfrac{a(a+b)\times(ab-b^2)}{a(a+b)\times(a^2+ab)} = \dfrac{ab-b^2}{a^2+ab}.$

## Examples XXIII.

(1.) Reduce to their lowest terms the following fractions :—

$$\frac{3x^2}{4xyz},\quad \frac{7aby}{14x^2y^2},\quad \frac{12a^2b^2}{14c^2d^2},\quad \frac{125a^4b^5c^6d^7}{350a^3b^7c^5d^6},\quad \frac{a-1}{a^2-1},\quad \frac{2a+2b}{3a^2-3b^2}.$$

$$\frac{x^4-1}{3x^2+3},\quad \frac{4a^2b-4ab^2}{4a^2b^2+4a^2b^4},\quad \frac{y^2+ay}{5by-2cy},\quad \frac{x^3+1}{x^2(x+1)^2},\quad \frac{7(a^2-1)}{14(a^3-1)}$$

$$\frac{9a^2b^2c^2}{4a^2bc+4ab^2c+4abc^2},\quad \frac{x^2-9x+14}{x^2+x-6},\quad \frac{x^3-x^2y+xy^2-y^3}{x^4+2x^2y^2+y^4}$$

$$\frac{x^3-3x-2}{2x^3+3x^2-1},\quad \frac{x^2+ax}{x^2+ax+bx+ab},\quad \frac{ax+ay+bx+by}{ax+ay-bx-by}$$

$$\frac{x^5-ax^4+a^4x-a^5}{x^5-a^5},\quad \frac{pqr(p+q+r)}{p^2+q^2+r^2+2pq+2pr+2qr},\quad \frac{m^5-mn^4}{m^3+mn^2}$$

$$\frac{a^2+2ab+b^2-1}{a^2-2b-b^2-1},\quad \frac{x^6-x}{x^5-x^4-x^3-x^2-x-2}.$$

**59.** To reduce a quantity of the form of an improper fraction to a mixed quantity.

### RULE.

*Divide as in ordinary Division, so long as the work can be continued; the quotient will be the integral part, the remainder the numerator, and the divisor the denominator of the fractional part.*

Reduce $\dfrac{a^3+x^3}{a-x}$ to a mixed quantity.

*Working.*

$$\begin{array}{r} a-x)\,a^3+x^3\,(x^2+ax+x^2 \\ \underline{a^3-a^2x\phantom{+x^3\,(x^2+ax+x^2}} \\ a^2x+x^3\phantom{\,(x^2+ax+x^2} \\ \underline{a^2x-ax^2\phantom{\,(x^2+ax+x^2}} \\ ax^2+x^3\phantom{+ax+x^2} \\ \underline{ax^2-x^3\phantom{+ax+x^2}} \\ \underline{2x^3\phantom{+ax+x^2}} \end{array}$$

$$\therefore\ \frac{a^3+x^3}{a-x} = a^2+ax+x^2+\frac{2x^3}{a-x}.$$

## Examples XXIV.

Reduce the following to mixed quantities:—

$$\frac{a+b}{a-b},\quad \frac{5x^2}{x-1},\quad \frac{a^2+b^2}{a-b},\quad \frac{x^2+2x+3}{x-7},\quad \frac{x^3-4}{x+1},\quad \frac{x^2+4xy+2y^2}{x+y}$$

$$\frac{x^2+y^2+z^2+xz+xy+yz}{x+y+z},\quad \frac{a^3-b^3}{a+b},\quad \frac{12x^3+9x^2y+28y^2(x+y)}{3x^2+7y^2}$$

$$\frac{(a+b)(a^2+b^2)}{(a-b)},\quad \frac{(a-b)^3}{a^2+b^2}.$$

## 60. To reduce Fractions to a common Denominator.

### RULE.

*Find the Least Common Multiple of all the denominators; this will be the common Denominator of all the Fractions. Divide this* L. C. M. *by the denominator of the first fraction, and multiply the first numerator by the quotient obtained; and proceed in the same way with every other fraction.*

Reduce $\frac{1}{x}$, $\frac{a}{x^2}$, $\frac{a^2}{x^3}$ to a common denominator.

*Working.* By inspection, L. C. M $= x^3$.

And $x^3 \div x = x^2$

$$\therefore\ \frac{1}{x} = \frac{x^2}{x^3}.$$

Also, $x^3 \div x^2 = x$

$$\therefore\ \frac{a}{x^2} = \frac{ax}{x^3}.$$

The last denominator is the same as the L. C. M. and need not be changed.

$$\therefore\ \frac{1}{x},\ \frac{a}{x^2},\ \frac{a^2}{x^3} = \frac{x^2}{x^3},\ \frac{ax}{x^3},\ \frac{a^2}{x^3}.$$

Reduce $\frac{a+x}{2(a-x)}$, $\frac{a-x}{3(a+x)}$, $\frac{2a^2}{3(a^2-x^2)}$ to a common denominator.

Here L. C. M. $= 6(a^2-x^2) = 6(a-x)(a+x)$

Now, $6(a^2-x^2) \div 2(a-x) = 3(a+x)$

$$\therefore\ \frac{a+x}{2(a-x)} = \frac{3(a+x)(a+x)}{6(a^2-x^2)} = \frac{3a^2+6ax+3x^2}{6(a^2-x^2)}$$

Again, $6(a^2-x^2) \div 3(a+x) = 2(a-x)$

$$\therefore \quad \frac{(a-x)}{3(a+x)} = \frac{2(a-x)(a-x)}{6(a^2-x^2)} = \frac{2a^2-4ax+2x^2}{6(a^2-x^2)}$$

$$6(a^2-x^2) \div 3(a^2-x^2) = 2$$

$$\therefore \quad \frac{2a^2}{3(a^2-x^2)} = \frac{2\times 2a^2}{6(a^2-x^2)} = \frac{4a^2}{6(a^2-x^2)}$$

## Examples XXV.

Reduce to a common denominator the following sets of fractions :—

$$\frac{a}{b},\ \frac{a^2}{b^2};\quad \frac{1}{2x},\ \frac{1}{3x^2},\ \frac{1}{4x^3};\quad \frac{b}{a},\ \frac{b^2}{ac},\ \frac{a^2}{bc},\ \frac{bc}{a^2};$$

$$\frac{a}{bc},\ \frac{b}{ac},\ \frac{c}{ab};\quad \frac{1}{abc},\ \frac{1}{abd},\ \frac{1}{bcd},\ \frac{1}{acd}$$

$$\frac{bc}{x^2},\ \frac{ac}{y^2},\ \frac{ab}{z^2};\quad \frac{1}{a},\ \frac{1}{b},\ \frac{1}{a+b},\ \frac{1}{a-b},\ \frac{x}{a+x},\ \frac{a}{a-x}$$

$$\frac{a+x}{2},\ \frac{a-x}{3},\ \frac{a^2-x^2}{4x};\quad \frac{a}{2},\ \frac{b}{3},\ \frac{c}{4},\ \frac{d}{5}$$

$$\frac{x-y}{xy},\ \frac{y-z}{yz},\ \frac{x-z}{xz};\quad \frac{1}{x-1},\ \frac{x}{x^2-1},\ \frac{x^2}{x^3-1}$$

$$\frac{a^2}{x^2\quad y^2},\ \frac{a^2}{x^2+y^2},\ \frac{a^4}{x^4-y^4};\quad \frac{xy}{2(x-y)},\ \frac{yz}{2(y-z)},\ \frac{xz}{2(x-z)}$$

$$\frac{p}{m-n},\ \frac{pq}{m^2-mn},\ \frac{pq^2}{m^3-m^2n};\quad \frac{x+y}{x^2+xy+y^2},\ \frac{x-y}{x^2-xy+y^2}$$

$$\frac{1}{(x-a)^3},\ \frac{1}{(x+a)^3},\ \frac{1}{(x^2-a^2)^2},\ \frac{1}{(x^2+a^2)^2};\quad \frac{x^4+a^4}{x^4-a^4},\ \frac{x^4-a^4}{x^4+a^4}.$$

$$\frac{x}{x+1},\ \frac{x^2}{x^2+1},\ \frac{x^3}{x^3+1},\ \frac{x^4}{x^4+1}$$

$$\frac{a}{2x-2a},\ \frac{2a^2}{3x^2-6ax+3a^2},\ \frac{3a^3}{4a^3-12a^2+12ax^2-4x^3}$$

### 61. To add or subtract Fractions.

RULE.

*Reduce them to a common denominator, then take the sum or difference of the numerators, and retain the same denominator.*

Add together $\dfrac{x}{a},\ \dfrac{x}{x},\ \dfrac{ax}{a^2-x^2}$.

L. C. M. $= ax(a^2-x^2)$.

$$\therefore\ \frac{x}{a}+\frac{a}{x}+\frac{ax}{a^2-x^2}=\frac{x^2(a^2-x^2)+a^2(a^2-x^2)+a^2x^2}{ax(a^2-x^2)}$$

$$=\frac{a^4+a^2x^2-x^4}{ax(a^2-x^2)}.$$

Subtract $\dfrac{x}{3(x^2-1)}$ from $\dfrac{1}{6(x+1)}$.

Here the L. C. M. $= 6(x^2-1)$

$$\therefore\ \frac{1}{6(x+1)}-\frac{x}{3(x^2-1)}=\frac{(x-1)-2x}{6(x^2-1)}$$

$$=\frac{-x-1}{6(x^2-1)}$$

$$=\frac{-(1+x)}{-6(1-x^2)}$$

$$=\frac{1}{6(1-x)}$$

## Examples XXVI.

(1.) Find the following sums :—

$$\frac{a}{2}+\frac{a}{3}+\frac{a}{4};\quad \frac{a}{5}+\frac{b}{10}+\frac{c}{10};\quad \frac{1}{a}+\frac{2}{b}+\frac{3}{b^4}+\frac{4}{a^2}$$

$$\frac{a+x}{a-x}+\frac{2a}{x}+\frac{3x}{a};\quad \frac{a}{x+a}+\frac{x+a}{a}+\frac{x^2-a^2}{x^2+a^2}+\frac{x^2+a^2}{x^2-a^2}$$

$$\frac{a+b}{a^2+ab+b^2}+\frac{a-b}{a^2-ab+b^2};\quad \frac{1}{a-b}+\frac{a^2}{a^3-b^3}+\frac{a}{a^2+ab+b^2}$$

$$\frac{y}{x}+\frac{2y^2}{x^2}+\frac{3y^3}{x^3}+\frac{4y^4}{x^4};\quad \frac{x+2}{x^2-1}+\frac{x-2}{x^2+1}+\frac{x^3}{x^4-1}$$

$$\frac{ax}{(x-a)^2}+\frac{a^2x^2}{(x-a)^4}+\frac{a^3x^3}{(x-a)^6};\quad \frac{x}{x+7}+\frac{x}{x+9}+1$$

$$\frac{x-1}{x+3}+\frac{2x+3}{x^2+5x+6};\quad \frac{a+b+c}{abc}+\frac{b+c+d}{bcd}+\frac{a+c+d}{acd}+\frac{a+b+d}{abd}$$

(2.) Find the following differences :—

$$\frac{x}{2}-\frac{x}{3};\quad \frac{4a}{b}-\frac{b}{4a};\quad \frac{x^2}{a^2}-\frac{x}{a};\quad \frac{1}{a-x}-\frac{1}{a+x}$$

$$\frac{a^2}{a^2-x^2}-\frac{a}{a+x};\quad \frac{x+1}{x^2+7x+12}-\frac{3}{x+4};\quad \frac{1}{x-4}-\frac{1}{x-3}$$

$$\frac{x}{x+6}-\frac{x^2+x-42}{x^2-x-42};\quad \frac{4x^3}{x^4+7x^2+10}-\frac{3x}{x^2+2}$$

$$\frac{a+1}{ax+1}-\frac{a^2+1}{a^2x^2-1};\quad \frac{3}{4(x-1)}-\frac{5}{6(x^3-1)}.$$

In some examples, Addition and Subtraction are combined, as in the following :—

Find the value of—

$$\frac{a}{a+x} - \frac{ax}{a^2-x^2} + \frac{x}{a-x}$$

*Working.* The L. C. M. $= a^2-x^2$,

$$\therefore \frac{a}{a+x} - \frac{ax}{a^2-x^2} + \frac{x}{a-x} = \frac{a(a-x)-ax+x(a+x)}{}$$

$$= \frac{a^2-ax+ax+ax+x^2}{a^2-x^2}$$

$$= \frac{a^2+ax+x^2}{a^2-x^2}.$$

(3.) In the same way, the following results may be simplified—

$$\frac{3}{x} + \frac{4}{x^2} - \frac{5}{x^3}; \quad \frac{a-b}{ab} - \frac{a-c}{ac} - \frac{c-b}{bc}$$

$$\frac{x}{x-a} - \frac{2x^2}{(x-a)^2} - \frac{x(x^2-a^2)}{(x-a)^3}.$$

$$\frac{1}{a+b+c+d} + \frac{2}{a+b-c-d} - \frac{3}{c+d-a-b}.$$

In this example, it should be observed that the two latter fractions have the same denominator, for, on multiplying the numerator and denominator of the fraction,

$$\frac{3}{c+d-a-b} \text{ by } -1 \text{ it becomes } \frac{-3}{a+b-c-d}.$$

$$\frac{3}{x+a} - \frac{2}{a+x} + \frac{1}{a^2-x^2} - \frac{3}{x^2-a^2}; \quad \frac{\overline{x+a}}{x-a} - \frac{\overline{x-a}}{x+a} - \frac{a^2-x^2}{a^2+x^2}$$

$$- \frac{a^2+x^2}{a^2-x^2}; \quad \frac{4a}{x^2+ax+a^2} - \frac{3x^2}{x^3-a^3} + \frac{4x^2}{a^3+x^3} - \frac{3x}{x^2-ax+a^2}$$

$$\frac{4x^3}{a-b} - \frac{a}{b-c} + \frac{4x^3}{b-a} + \frac{b}{a-c} - \frac{b}{c-a} - \frac{a}{c-b}$$

$$\frac{x^2-a^2}{3(x^2+ax+a^2)} + \frac{(a^2-x^2)}{5(a^3-x^3)} - \frac{a}{a-x}.$$

**62.** To multiply Fractions together, observe the following

RULE.

*Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

This Rule may be proved as follows :—

Let $$\frac{a}{b} = x, \quad (1.)$$

And $$\frac{c}{d} = y, \quad (2.)$$

Multiplying both sides of (1) by $b$, we have—

$$a = bx$$

Similarly, $$c = dy$$

Multiplying these last two equations together, we have—

$$ac = bdxy$$

And dividing both sides of this result by $bd$,

$$\frac{ac}{bd} = xy$$

or, $$x \times y = \frac{ac}{bd}.$$

That is, $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{\text{product of the Numerators.}}{\text{product of the Denominators.}}$$

*Note.*—In multiplying fractions together, the work should always be shortened by cancelling common factors in the numerator and denominator.

(1.) Multiply together $\frac{4a}{5b}$ and $\frac{3a^2x}{11by^2}$.

$$\frac{4a}{5b} \times \frac{3a^2x}{11by^2} = \frac{12a^3x}{55b^2y^2}.$$

(2.) Multiply together $\frac{2x}{3(x-y)}$ and $\frac{9(x^2-y^2)}{11}$.

$$\frac{2x}{3(x-y)} \times \frac{9(x^2-y^2)}{11(x^2+y^2)} = \frac{2x \times 9(x-y)(x+y)}{3(x-y) \times 11(x^2+y^2)} = \frac{6x(x+y)}{11(x^2+y^2)}.$$

## Examples XXVII.

(3.) Find the products of the following :—

$$\frac{a^3}{b} \text{ and } \frac{b^3}{c^3}; \quad \frac{c^2}{2d}, \ \frac{2b^2}{3c}, \text{ and } \frac{4c^2}{5abd}$$

$$\frac{x+a}{x^2-ax+a^2} \text{ and } \frac{x^2-ax+a^2}{x+a}; \quad \frac{a^2}{cx-ac} \text{ and } \frac{b}{x+a}$$

$$\frac{7a^2}{4x-8} \text{ and } \frac{7x-14}{14a^2y^2}; \quad \frac{4x}{12x^2-5x-2} \text{ and } \frac{4x+1}{7ax^3}$$

$$\frac{9x+1}{6x^2+5x-56} \text{ and } \frac{3x-8}{27x^2+48x+5}$$

$$\frac{2x^3+7}{99x^5-81x^3-121x^2+99} \text{ and } \frac{11x^2-9}{6x^5-16x^3+21x^2-56}$$

$$\frac{x-a}{x^2-4a^2},\ \frac{x+2a}{x^3-a^3}, \text{ and } \frac{x-2a}{x^2-ax+a^2}.$$

(4.) Multiply together $1+\frac{a}{1-a}$ and $2-\frac{1-a}{1+a}$.

These expressions must be reduced to a fractional form at first.

Then $$\left(1+\frac{a}{1-a}\right)\times\left(2+\frac{1-a}{1+a}\right) = \frac{1}{1-a}\times\frac{3+a}{1+a}$$
$$= \frac{3+a}{1-a^2}.$$

(5.) Find the following products :—

$$\left(2+\frac{2a}{a-b}\right)\times\left(1-\frac{a-b}{a+b}\right);\quad \frac{a^4-ab^3}{a^3b+b^4}\times\frac{a^2+2ab+b^2}{a^3-2a^2b+ab^2}.$$

$$\frac{4\,(a^2-ab)}{a^2b+2ab^2+b^3}\times\frac{a^2-b^2}{12ab};\quad \frac{(a+x)^3}{x^3+y^3}\times\frac{x^3-x^2y+xy^2}{a^2+2ax+x^2}.$$

$$\frac{6x^3}{a^3+x^3}\times\frac{a+x}{ax}\times\left(1-\frac{ax}{a^2+x^2}\right)$$

$$\left(\frac{a}{1+a}+\frac{1-a}{a}\right)\times\left(\frac{a}{1-a}+\frac{1-a}{2}\right);\quad \frac{p+q}{3\,(p-q)}\times\frac{p^2+pq-2q^2}{p^2+pq}$$

$$\frac{6p^4+p^2q^2-12q^4}{4p^4-p^3q-20p^2q^2+5pq^3}\times\frac{3p^4-19p^2q^2+20q^4}{8p^4-2p^3q+12p^2q^2-3pq^3}\times$$
$$\frac{p^2\,(4p-q)^2}{(3p^2-4q^2)^2};\quad \frac{a-2b}{6a^2-ab-b^2}\times\frac{6a+2b}{2a^2-5ab+2b^2}$$

$$\left(1+\frac{x}{1-x}\right)\times\left(1-\frac{x}{1+x}\right)\times\frac{1-x^2}{1+x^2};\quad \left(\frac{1}{1-x}+\frac{x}{1+x}\right)$$
$$\left(\frac{1}{1-x}-\frac{x}{1+x}\right)\quad\left(\frac{1+x}{1-x}\right)^2;\quad \left(\frac{a-b}{a+b}-\frac{ab}{a^2-b^2}\right)\times$$
$$\frac{a^2+2ab-3b^2}{a-3b}\times\frac{a^2+b^2}{(a+3b)\,(a^2-3ab+b^2)}.$$

## 63. To divide one Fraction by another.

RULE.

*Invert the divisor, and then multiply the Fractions together.*

This rule may be proved as follows :—

*Let* $$\frac{a}{b} = x$$

And $$\frac{c}{d} = y$$

$$\therefore \quad a = bx, \quad (1)$$

And $$c = dy, \quad (2)$$

Multiply (1) by $d$, and (2) by $b$, we have—

$$ad = bdx$$

$$bc = bdy$$

$$\therefore \quad \frac{ad}{bc} = \frac{bdx}{bdy}$$

or, $$\frac{ad}{bc} = \frac{x}{y}.$$

That is, $$x \div y = \frac{a}{b} \times \frac{d}{c}.$$

or, $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

### Examples.

(1.) Divide $\frac{4a}{3b}$ by $\frac{8ab^2}{9xy}$.

$$\frac{4a}{3b} \div \frac{8ab^2}{9xy} = \frac{4a}{3b} \times \frac{9xy}{8ab^2} \text{ by the Rule.}$$

$$= \frac{3xy}{2b^3} \text{ by omitting common factors.}$$

(2.) Divide $\frac{a}{a+x}$ by $\frac{3b}{a-x}$.

$$\frac{a}{a+x} \div \frac{3b}{a-x} = \frac{a}{a+x} \times \frac{a-x}{3b}$$

$$= \frac{a^2-ax}{3ab+3bx}.$$

(3.) Divide $\frac{a}{4} + \frac{1}{2}$ by $\frac{a^2}{8} - \frac{1}{2}$.

Now, $$\frac{a}{4} + \frac{1}{2} = \frac{a+2}{4}$$

And, $$\frac{a^2}{8} - \frac{1}{2} = \frac{a^2-4}{8}$$

And, $$\frac{a+2}{4} \div \frac{a^2-4}{8} = \frac{a+2}{4} \times \frac{8}{a^2-4}$$

$$= \frac{a+2}{4} \times \frac{8}{(a+2)(a-2)}$$

$$= \frac{2}{a-2}.$$

*Note.*—If mixed quantities are to be divided, they should be reduced to improper fractions before the Rule is applied.

### Examples XXVIII.

(1.) Divide $\frac{a}{b}$ by $\frac{2a}{b}$; $\frac{a^2}{b^2}$ by $\frac{b}{a}$; $\frac{5x}{6y}$ by $\frac{3}{4}$.

$-\frac{7ax}{4by}$ by $-\frac{4ax}{7y^2}$; $\frac{3}{5}x$ by $\frac{4}{5}y$; $\frac{3}{4}x^2y^2$ by $\frac{4}{5}x^3y$

$-\frac{9a}{11b}$ by $\frac{2a^2}{3b^2}$; $\frac{x^2y^3z^4}{ab^2c^3}$ by $\frac{xy^2z^3}{a^2b^3c^4}$; $\frac{4x^2}{3y^2}$ by $-\frac{4y^2}{9x^2}$

$-\frac{2a^2b^4c^6d^8}{9p^4q^8r^{12}}$ by $\frac{a^4b^3c^2d}{3p^2q^4r^6}$; $\frac{\frac{3}{5}xy}{\frac{5}{6}ab}$ by $\frac{\frac{3}{4}x^2y^2}{\frac{7}{12}a^3b^3}$.

(2.) Divide $\frac{4a+6b}{7a^2+14b^2}$ by $\frac{2x}{7y}$.

*Working.* $\frac{4a+6b}{7a^2+14b^2} \div \frac{2x}{7y} = \frac{(2a+3b)}{(a^2+2b^2)} \times \frac{y}{x}$

$= \frac{2a+3b}{a^2+2b^2} \times \frac{y}{x} = \frac{2ay+3by}{a^2x+2b^2x}$.

Divide $\frac{a^3+a^2b}{b^3-ab^2}$ by $\frac{a^2}{b^2}$; $\frac{a+b+c}{a+b-c}$ by $\frac{x}{2y}$,

$\frac{a^2x^2+abxy}{p^2x+pqy}$ by $-\frac{ax}{pq}$; $\frac{a+2b+3c}{a^3-2a^2b+3a^2c}$ by $\frac{2b}{3a^2}$

$\frac{x^2+1}{x}$ by $\frac{2}{x}$; $\frac{4x^3}{1-x^2}$ by $\frac{x^2}{4}$; $-\frac{a^2-x^2}{ax}$ by $\frac{ab}{xy}$.

(3.) Divide $\frac{a^2+b^2}{3a+3b}$ by $\frac{4a^4+4a^2b^2}{9ab+9b^2}$.

*Working.* $\frac{a^2+b^2}{3a+3b} \div \frac{4a^4+4a^2b^2}{9ab+9b^2} = \frac{a^2+b^2}{3(a+b)} \times \frac{9b(a+b)}{4a^2(a^2+b^2)}$

$= \frac{3b}{4a^2}$ by omitting from the Numerator and Denominator the common factors $(a^2+b^2)$, $(a+b)$, and 3.

*Note.*—All the examples in Division should be worked out in the same form as (2) and (3). The first step should always be to break the numerators and denominators into their simple factors.

(4.) Find the following quotients:—

$\frac{8xy+8y^2}{9ab-9b^2} \div \frac{6x^2+6xy}{a^2-ab}$; $\frac{3abc+6a^2c}{2b^2c-5abc} \div \frac{b+2a}{3b^2c^2}$; $\frac{1}{x-1} \div \frac{1}{x+1}$

$\frac{3}{x+2} \div \frac{6}{x+4}$; $\frac{x^3+1}{x-1} \div \frac{x+1}{x^3-1}$; $\frac{(a-b)(a-c)}{(a+b)(a+c)} \div$

$$\frac{(a^2-b^2)(a^2-c^2)}{(a+b)^2(a+c)^2};\quad \frac{x^2+4y^2}{x^3+8y^3}\div\frac{x^2-4y^2}{x^2-2xy+4y^2}$$

$$\frac{x^2+1}{x^4+1}\div\frac{3}{x^3+2x^2-x-2};\quad \frac{4x^2+32x+28}{7x^3-63x}\div\frac{x+1}{x+3}$$

$$\frac{x^2-4x-5}{2x^2+4x-6}\div\frac{x^2-6x+5}{3x^2+12x+9};\quad \frac{x^6-1}{x^6+4x^3+4}\div\frac{x^2-1}{x^6-4}.$$

(5.) Divide $\left(1-\frac{2x}{1+2x}\right)$ by $\left(3+\frac{2-x}{1+2x}\right)$.

*Working.* $1-\frac{2x}{1+2x}=\frac{1+2x-2x}{1+2x}=\frac{1}{1+2x}.$

And $3+\frac{2-x}{1+2x}=\frac{3+6x+2-x}{1+2x}=\frac{5+5x}{1+2x}=\frac{5(1+x)}{1+2x}$

$$\therefore\quad \frac{1}{1+2x}\div\frac{5(1+x)}{1+2x}=\frac{1}{1+2x}\times\frac{1+2x}{5(1+x)}=\frac{1}{5(1+x)}.$$

*Note.*—Mixed quantities must be reduced to the form of improper fractions before they can be divided.

Find the following quotients:—

$$\left(1-\frac{2a}{1+a}\right)\div\frac{a}{6};\quad \left(a-\frac{1+a}{1-a}\right)\div\frac{1}{a^2-a}$$

$$\left(x-\frac{1}{2}\right)\div\left(1-\frac{1}{4x^2}\right);\quad \left(a^2+\frac{a}{x}\right)\div\left(\frac{4+ax}{3ax}+1\right)$$

$$\left(\frac{1}{4}-x^2\right)\div\left(\frac{2+5x}{1+3x}-1\right);\quad \left(1-\frac{x^2}{y^2}\right)\div\left(\frac{2x}{x^2+y^2}-1\right).$$

Find the following quotients:—

$$\frac{a^2}{b^2}\div\frac{a}{b};\quad \frac{3a^2}{4b^3}\div\frac{6a^3}{4b^2};\quad \frac{ax^2}{by^2}\div\frac{2abx}{3cdy};\quad \frac{7a+7b}{9a-9x}\div\frac{14a^2-14b^2}{3a^2-3x^2}$$

$$\frac{x^2-ax+a^2}{x^3-a^3}\div\frac{x^3+a^3}{x^2+ax+a^2};\quad \left(1+\frac{x}{a}\right)\div\left(1-\frac{x^2}{a^2}\right)$$

$$\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\div\left(x+\frac{1}{x^2}\right);\quad \left(\frac{a^3}{b^3}-\frac{b^3}{a^3}\right)\div\frac{a^4+a^2b^2+b^4}{a^4b^4c^4}$$

$$\frac{a^4+10a^2b^2+21b^4}{a^4-3a^2b^2-70b^4}\div\left(\frac{a^6-49a^2b^4}{(a^2+7b^2)^2}\times\frac{3a^3+21ab^2}{7a^3-70ab^2}\right).$$

## Miscellaneous Examples in Fractions.

(1.) Find the sum, difference, product, and quotient of $\frac{1}{a}$ and $\frac{1}{b}$; also, find the difference between $\frac{1}{a+b}$ and $\frac{1}{a}+\frac{1}{b}$.

(2.) Reduce $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$ to a simple fraction,

and find which is the greatest and which the least of the four fractions—

$$\frac{a+b}{a^3-b^3},\quad \frac{1}{a^2-b^2},\quad \frac{1}{a^2+ab+b^2},\quad \frac{1}{a^2+b^2}.$$

(3.) Find the continued product of the following fractions:—

$$\left(1+\frac{x}{y-x}\right),\quad \left(1-\frac{x}{x-y}\right),\quad \left(\frac{1}{x^2}-\frac{1}{y^2}\right),\quad \left(1-\frac{x}{x+y}\right).$$

(4.) Show that—

$$b^2-\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\frac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}.$$

(5.) Add together the following fractions:—

$$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$$

(6.) Reduce to their lowest terms the following fractions:—

$$\frac{a^3+ab^2-b^3-a^2b}{4a^4-4a^3b-2a^2b^2+2ab^3} \text{ and } \frac{ac^m-bc^{m+1}}{a^2bc-b^3c^3}.$$

(7.) Reduce to their simplest forms the following:—

$$\frac{1}{2x+y}+\frac{1}{2x-y}-\frac{3x}{4x^2-y^2}$$

$$\frac{1}{x}+\frac{x-5}{x(x+1)}+\frac{x^2+8}{x(x+1)(x+2)}$$

$$\frac{1}{x-1}-\frac{x+1}{x^2+x+1}-\frac{x}{x^3-1}.$$

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## ON EQUATIONS.

**64.** An equation is a statement of the equality of two different Algebraical expressions.

The two expressions are separated from each other by the sign of equality.

Thus, $x+1 = 3x-7$

*is an* equation.

The expression on the left hand of the sign of equality is called the first side; that on the right hand, the second side of the equation.

Again, $x^3-3x^2+4x-7 = 0$

is an equation whose second side is zero.

When both sides of an equation consist of the same expression, or when they can both be reduced to the same expression, the equality of the two is called an identity.

Thus $4x+4 = 4(x+1)$ is an identity.

Also, $x-1+x-4 = 2x-7+2$

And $x^2-a^2 = (x-a)(x+a)$ are identities.

An identity is always true, whatever value of the letters may be taken.

One or more of the letters in an equation always represents an unknown quantity; the letters $x\,y\,z$ are generally used to denote the unknown quantities.

$$x+3 = 13-4x$$

is an equation of *one* unknown quantity.

Again, $x-4y = 7y-10x+22$

is an equation of two unknown quantities.

The equation $x+3 = 13-4x$ may be thus explained:—

"A certain number when increased by 3, gives the same result as when four times that number is subtracted from 13."

This number is at present unknown, and is denoted by $x$.

If 2 be substituted for $x$ in the above equation, the first side becomes $2+3$ or 5, and the second side becomes $13-8$ or 5; that is, the equation is true when $x = 2$. And it can easily be seen that the equation is not true for any other value of $x$ than 2; for if a greater value than 2 be written for $x$, the first side will be greater than the second, and if a less value than 2 be written, the first side will be less than the second.

**Def.** *That value of the unknown quantity which makes both sides of the equation equal, is called the root of the equation.*

When this root is known the equation is said to be solved.

Thus 2 is the root of the above equation.

$$x+3 = 13-4x.$$

**65.** *A simple equation of one unknown quantity is that in which only the first power of the unknown quantity occurs.*

Before giving examples of the solution of simple equations it will be necessary to consider some of the processes employed.

Take the equation—

$$\frac{4x-7}{5} + 2x = \frac{3x+17}{2} - 6 \quad (a).$$

If both sides be multiplied by 10, the equality will not be destroyed, that is—

$$10 \times \frac{4x-7}{5} + 20x = 10 \times \frac{3x+17}{2} - 60.$$

$$\text{Or,} \quad 2(4x-7) + 20x = 5(3x+17) - 60.$$

$$\text{Or,} \quad 8x-14+20x = 15x+85 - 60. \quad (b).$$

$$\therefore \quad 28x-14 = 15x+25. \quad (b).$$

Let the same quantity $(14-15x)$ be added to both sides, the equality will not be destroyed. The equation now becomes—

$$28x-14+14-15x = 15x+25+14-15x$$

$$28x-15x = 25+14 \quad (c).$$

$$13x = 39.$$

If now we divide both sides of this by 13, the equation will still be true, and the result is—

$$x = 3.$$

The equation is now solved, and 3 is its root.

**66.** The axioms that have been assumed in the preceding process of solution are these.

i. *Both sides of an equation may be multiplied or divided by the same quantity, without destroying their equality.*

This is obviously true, for if two quantities be equal to one another, any the same multiples, or the same parts of them must be equal to one another.

ii. *The same quantity may be added to, or subtracted from, both sides of an equation, without destroying their equality.*

This is also true, for, if equals be added to, or taken *from,* equals, the wholes or the remainders are equal.

*Note.*—To shorten the process of solution, quantities are not added to, or subtracted from, both sides of an equation, but are transposed from one side of it to the other, according to a Rule which will now be explained.

Equation (*b*) in the preceding process is—

$$28x-14 = 15x+35.$$

And Equation (*c*) is—

$$28x-15x = 14+35.$$

And these are both the same equation as was shown.

This change in the form of the equation was made by adding an equal quantity to both sides. It will be observed that two of the quantities, viz., 14 and $15x$, have changed *sides* in the equation; also, that they have changed *signs*. In fact, (*c*) might have been formed at once from (*b*) by writing 14 on the second side with a $+$ sign, and $15x$ on the first side with a $-$ sign.

The following is the RULE of Transposition:—

iii. *When, in an equation, any quantity changes its side, it must also change its sign.*

This is also an axiom; for when two quantities are equal, if any positive quantity be removed from one, an equal negative quantity must be added to the other, in order that they may continue equal: also, if any negative quantity be removed from one, an equal positive quantity must be added to the other.

67. From the preceding process, the following RULES may be derived for the solution of a simple equation of one unknown quantity.

(a.) *If there be fractions in the equation, multiply both sides by the* L. C. M. *of all the denominators.*

(b.) *Remove brackets, and simplify each side by collecting like quantities.*

(c.) *Apply the Rule of Transposition to bring all the unknown terms to the first side, and all other terms to the second side, and simplify each side.*

(d.) *Divide both sides by the coefficient of* x, *and the result will be the root of the equation.*

Solve the equation $\frac{3x-4}{5} = \frac{2x+9}{15}$.

$$\begin{aligned} &\text{Clearing of Fractions} & 3(3x-4) &= 2x+9 \\ &\qquad\qquad\therefore & 9x-12 &= 2x+9 \\ &\text{Transposing} & 9x-2x &= 12+9 \\ &\qquad\qquad\therefore & 7x &= 21 \\ &\text{And} & x &= 3. \end{aligned}$$

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$$\frac{2}{5x} - \frac{7}{2x} + \frac{13}{30x} + \frac{4}{3} = 0.$$

L. C. M. $= 30x$.

$$\begin{aligned} &\text{Clearing of Fractions} & 12-105+13+40x &= 0 \\ &\text{By Transposition } \therefore & 40x &= 105-13-12 \\ &\text{Or,} & 40x &= 80 \\ &\qquad\qquad\therefore & x &= 2. \end{aligned}$$

## Examples XXIX.

Solve the following equations :—

(1.) $7x+10 = 12x-5$

(2.) $14x = 3(5x-2) - 1$

(3.) $3x+10 = 5x-6$

(4.) $4x-7 = 8-x$

(5.) $11+x = 12x$

(6.) $14-2(x-1) = 5(x-1)$

(7.) $4(x-1) - 20(x-2) = 2(3-x)$

(8.) $17-3x+4 = 8x-(x-1)$

(9.) $(x-1)+(x-2)+(x-3) = 0$

(10.) $7(x+1) = 13(x-1)$

(11.) $15-11+4 = x-3x+10x$

(12.) $5(x+1) - 7(x-1) = 0$; $3x+5 = 10x-16$; $2x+11 = 7x-19$; $20-(4x-8) = 5(x+2)$; $2(x-12)+3(x-10) = 6$; $2x-7 = 7+30x$; $x+3(x-2) = 2(x-1)+4$; $12x+7(x-1) = 42(x-4)$; $3\{x-2(x-1)\} = 4(5+x)$; $3x+x-2-3(x-1)+4 = 0$; $2(x+a+b) = (x+a-b)-(x-a+b)$; $a(x+b+c)+b(x+a+c)+c(x+a+b) = 0$; $(x-a)(x-b) = (x-c)(x-d)$; $x^2 = (x-1)(x-2)$; $30(2x+1) - 12(3-x) = 138$; $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 2\frac{1}{6}$.

(13.) $3(x-1)+4(x-2)+5(x-3)+2 = 0.$

(14.) $7(x-2)+3 = 2x-3-(x-4)$

(15.) $(x+12)-(x-12) = 4x$

(16.) $\frac{x}{5} = \frac{x}{6} + 3$; (17.) $\frac{x}{2} + \frac{x}{8} + \frac{x}{4} = 1\frac{1}{12}$

(18.) $4\left(x-\frac{1}{7}\right) - 3\left(x+\frac{1}{5}\right) + \frac{111}{35} = 0$

(19.) $\frac{x-1}{2} + \frac{x+1}{3} = \frac{4-x}{6} + \frac{x}{6}$

(20.) $\frac{x}{2} + \frac{x}{4} = 3$; (21.) $x - \frac{x}{6} = 6$

(22.) $\frac{3}{4}x - 9 = \frac{1}{4}x + 7$; (23.) $x - 2\frac{1}{4} = 11\frac{3}{4} - x$

(24.) $\frac{x}{2} - \frac{x}{4} - \frac{3}{4} = \frac{x}{3} - 1\frac{1}{3}$; (25.) $\frac{x}{7} + \frac{x}{12} - \frac{x}{2} = -1\frac{11}{12}$

(26.) $\frac{5x}{12} - \frac{3x}{8} + \frac{4x}{7} = \frac{5x}{14} + 3\frac{1}{6}$

(27.) $\frac{3}{4}(x+2) + \frac{1}{2}(x+3) = \frac{3}{2}(x+1)$

(28.) $\frac{1}{2}(x-2) + \frac{1}{3}(x-3) + \frac{1}{4}(x-4) - \frac{3}{5}\left(x+\frac{1}{6}\right) = 0$

(29.) $\frac{x+7}{2} - \frac{x+9}{3} + \frac{5x-7}{6} = 0$

(30.) $\frac{3x-4}{2} - \frac{2x-3}{4} + \frac{x-2}{5} = \frac{3}{20}$

(31.) $\frac{7x-6}{12} - \frac{x}{2} = 4\frac{1}{12}$

(32.) $\frac{2x-7}{11} + \frac{3x+1}{7} + \frac{x-8}{3} = 5$ (35.) $6x = 7 - \frac{1-4x}{3}$

(36.) $\frac{x}{7} - \frac{x}{2} = \frac{1-x}{12} - 2$

$3x\left(\frac{1}{3} - \frac{1}{x}\right) + \frac{5}{2}(2x+1) - 11\frac{1}{2} = 0$

(37.) $x - \frac{1}{2} = 9\frac{1}{2} - 4\frac{1}{2}\cdot x$; $12\frac{3}{4} - \frac{x}{2} = 2x - 9\frac{3}{4}$

(38.) $\frac{2x}{7} + \frac{x-1}{6} = x - 4$; $\frac{x+1}{2} - \frac{x+1}{3} = \frac{5}{6}$

(39.) $x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7$; $\frac{x+1}{2} = 14$

(40.) $\frac{x+1}{2} + \frac{x+1}{3} = 16 - \frac{5x+1}{4}$

(41.) $\frac{x+3}{2} - \frac{11-x}{5} = \frac{3x-1}{20} + 31$

(42.) $\frac{9x+1}{2} - \left(x - \frac{x-2}{7}\right) = 37$

$\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$

$\frac{a}{x} + \frac{b}{x} + \frac{c}{x} = d$; $\frac{1}{x} + \frac{2}{x} + \frac{5}{x} = 4$

$\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{7}{3}$; $\frac{ax-c}{2} + \frac{c}{3} = \frac{cx}{4} - \frac{bx+d}{5}$

(43.) $\frac{3x-4}{2x-7} - \frac{9x+1}{6x-2} = \frac{77}{(2x-7)(6x-2)}$

*Working.* L. C. M. $= (2x-7)(6x-2)$.

Clearing of fractions, we have—

$$(3x-4)(6x-2) - (9x+1)(2x-7) = 77$$

Or, $18x^2 - 30x+8 - 18x^2+61x+7 = 77$

Or, $61x-30x = 77-15$

$\therefore$ $31x = 62$

And $x = 2.$

(44.) $\frac{1}{x-1} + \frac{2}{x-2} = \frac{3}{x+2}$

(45.) $\frac{x+19}{x+12} = \frac{3x+4}{2x+1} - \frac{1}{2}$

(46.) $\frac{2x+4}{5} - \frac{x+2}{x-5} = \frac{2(x+1)}{5}$

(47.) $\frac{8}{x+1} - \frac{10+3x}{x+3} = \frac{5+x}{1+x} - \frac{5x}{3(x+3)} - \frac{7}{3}$

(48.) $\frac{1}{2x+c} + \frac{1}{2x-c} - \frac{3x-7}{4x^2-c^2} = 0$

(49.) $\frac{a}{x-3a} + \frac{a}{x+3a} = \frac{5a^2}{2(x^2-9a^2)}$

(50.) $1+ \frac{x}{x+a} = \frac{x^2}{(x+a)^2} + \frac{x^3+4ax^2+5a^3}{(x+a)^3}$

(51.) $\frac{2x-7}{2x-8} - \frac{3x+8}{3x+9} - \frac{x-3}{x^2-x-12} = 0$

(52.) $\frac{3}{x-7} + \frac{4}{x-6} = \frac{4x}{x^2-9x+14} + \frac{5x+3}{x^2-8x-12}$

(53.) $\frac{ax+bx+cx}{a+c} = x+d$; $\frac{70}{3x+5} = \frac{96}{5x-2}$

(54.) $\frac{6x^2}{x-2} = \frac{5x^2}{x-3}$; $\frac{x}{a} + \frac{c-x}{b} = 1$

(55.) $\frac{4x-34}{17} - \frac{258-5x}{3} = \frac{69-x}{2}$

(56.) $\frac{ax+b^2}{a^2+b} = \frac{a^2+bx}{a+b^2}$; $(1+x)\left(1+\frac{1}{1+x}\right) = 0$

(57.) $(1+x)\left(1+\frac{1}{1+x}\right) - (1-x)\left(1-\frac{1}{1-x}\right) = 6$

(58.) $(1-x)\left(1+\frac{1+x}{1-x}\right) + (1+x)\left(1-\frac{1-x}{1+x}\right) = 0.$

**68.** Equations are applied to the solution of Problems of various kinds, which cannot be solved by Arithmetic. Some examples of these will now be given. It will be necessary to bear in mind that the statement of a Problem in ordinary language can be translated into Algebraical language, and that this Algebraical language takes the form of an equation. When the equation is formed, it must be solved by the methods already given, and the solution of the problem is known.

### Examples.

(1.) A number is doubled, and ten is added to the double, and the whole sum is 56. Find the number.

Let $x$ = the number to be found.

$\therefore 2x$ = the double of it,

and $2x+10$ = the whole sum;

but $56$ = the whole sum.

$\therefore 2x+10 = 56.$

Then $2x = 46,$

and $x = 23,$ the number required.

The following Rules should be carefully attended to in the Solution of Problems involving equations of one unknown quantity.

(a.) *Let* x *be taken to denote the unknown quantity, or quantity required to be found.*

(b.) *Find expressions for all the quantities which occur in the statement of the problem.*

(c.) *Perform all the operations directed by the question upon these quantities.*

(d.) *Find two different expressions for the same thing, and put these equal to one another.*

(e.) *Solve the equation thus formed, and the answer to the problem will be known.*

(2.) Divide 57 into 2 parts, such that 3 times the less part subtracted from half the greater part shall leave 25.

Let $x$ = the greater part,
Then $57-x$ = the less part.
And $3(57-x)$ = three times the less part,
and $\frac{x}{2}$ = half the greater part.

$\therefore \frac{x}{2} - 3(57-x)$ = the difference between half the greater part, and three times the less.
Also 25 = the difference between half the greater part and three times the less.

$$\therefore \frac{x}{2} - 3(57-x) = 25.$$

Working out this, we find

$x$ = 56 the greater part,
and $57-x$ = $57-56=1$ the less part.

(3.) A grocer mixes sugar at $4\frac{1}{2}d.$ per lb. with sugar at $6\frac{1}{2}d.$ per lb. The value of the mixture is $5d.$ per lb., and the weight of the mixture is 480 lbs. Find the number of lbs. of each kind of sugar.

Let $x$ = the number of lbs. of sugar at $4\frac{1}{2}d.$ per lb.
$\therefore 480-x$ = " " " $6\frac{1}{2}d.$ "
And $\frac{9}{2}x$ = the value of 1st kind of sugar in pence.
$\frac{13}{2}(480-x)$ = " 2nd " "
$\therefore \frac{9}{2}x+\frac{13}{2}(480-x)$ = the whole value of the mixture.
Also $480\times 5$ = the whole value of the mixture.

$$\therefore \frac{9}{2}x+\frac{13}{2}(480-x) = 5\times 480$$

Solving this question, we find

$x$ = 360 the number of lbs. of the first kind;
and $480 - x$ = $480-360=120$ the number of lbs. of the second kind.

(4.) A number consists of three figures, in which the hundreds' figure is one less than the tens' figure, and the units' figure is one more than the tens' figure; and if 14 be added to the number, a new number is formed in which the units' figure is twice as great as the tens' figure and the tens' figure is twice as great as the hundreds' figure. Find the *number.*

Let $x$ = the hundreds' figure of the first number;
then $x+1$ = the tens' figure " " "
and $x+2$ = the units' figure " " "

$\therefore 100x+10(x+1)+(x+2)$ = the first number;
and $100x+10(x+1)+(x+2)+14$ = the second number.

Again $x$ = the hundreds' figure of the second number;
$2x$ = the tens' figure " " "
$4x$ = the units' figure " " "

$\therefore 100x+10\times 2x+4x$ = the second number.

$$\therefore 100x+10(x+1)+(x+2)+14 = 100x+10\times 2x+4x.$$

Solving this, we find

$x = 2$ the hundreds' digit of first number.
$x+1 = 3$ the tens' digit.
$x+2 = 4$ the units' digit.

$\therefore 234$ = the first number.
and $234+14 = 248$ the second number.

## Examples XXX.

### 69. *Easy Problems in Simple Equations of one Unknown Quantity.*

(1.) If a number be doubled, and six be added to the double, the whole sum is 20: find the number.

(2.) Find a number which is as much above 17 as it is below 33.

(3.) Divide 7 shillings between two boys, so that one shall have sixpence more than the other.

(4.) The united ages of three brothers are 37 years; the eldest is two years older than the second, and the second three years older than the youngest: find the age of each.

(5.) John starts at six o'clock in the morning on a journey, walking at the rate of 3½ miles an hour; William starts after him at 8 o'clock, walking at the rate of 4½ miles an hour: find where and when the latter will overtake the former.

(6.) The difference in the age of two persons is 16 years, and one is three times as old as the other: find the age of both.

(7.) How much tea at 3*s*. 9*d*. per lb. must be mixed with 50 lbs. of tea at 4*s*. 6*d*. per lb., in order that the mixture may be worth 4*s*. per lb.?

(8.) A cask of spirits is worth 12*s*. 6*d*. a gallon, but if 4 gallons of water be added to it, the price will be only 11*s*. 6*d*. a gallon: find the number of gallons in the cask.

(9.) A regiment on a forced march, marches 2 miles more every day than it marched on the preceding day; at the end of six days it is 138 miles from the starting-point: find the distance marched on each day.

(10.) Divide 25 shillings among 4 brothers, so that the three youngest may have equal shares, and the eldest may have as much as two of the others.

(11.) There are three fields, the largest of which is twice as large as the smallest, and the largest and smallest together are twice as large as the other, and all the fields contain 18 acres: find the number of acres in each field.

(12.) At a contested election, 1793 votes are polled, and the defeated candidate is left in a minority of 313: find the number of votes for each candidate.

(13.) A gentleman contracts with a builder to finish his house in 30 days; the contract is for £800, but the builder is to forfeit £5 for every day beyond the 30 days that it remains unfinished; he receives only £780: find the time at which the house was finished.

(14.) If one man can mow a field of grass in 7 days, and another can mow it in 5 days: in what time could both, working together, cut the field?

*Working.* Let $W$ denote the number of acres in the field
Let $x$ denote the required number of days.

Then, $\frac{W}{7}$ = number of acres mown by the first man in one day

And $\frac{W}{5}$ = ,, ,, second man ,,

And $\frac{W}{x}$ = ,, ,, both men ,,

$$\therefore \quad \frac{W}{x} = \frac{W}{5} + \frac{W}{7}.$$

Dividing both sides of this equation by $W$, it becomes—

$$\frac{1}{x} = \tfrac{1}{5} + \tfrac{1}{7}$$

Or, $$\frac{1}{x} = \tfrac{12}{35}$$

$$\therefore \quad x = \tfrac{35}{12}$$

$= 2\tfrac{11}{12}$ the number of days required.

(15.) A citadel is garrisoned by 800 men, who are provisioned for 80 days; at the end of 20 days they are joined by a regiment of 1000 men, who are provisioned for 120 days: how long will their provisions last after they join?

*Working.* Let $p$ = one man's daily provision

And $x$ = the required number of days

Now, $800 \times 60p$ = the quantity of provisions in the garrison when the reinforcement arrives

And $120 \times 1000p$ = the quantity of provisions brought by the reinforcement

$\therefore$ $48{,}000p \times 120{,}000p$ = the whole of the provisions when the reinforcement has arrived

But $(800+1000)x.p$ = the provisions eaten in $x$ days

$$\therefore \; (800+1000)x.p = (48{,}000+120{,}000)p$$

$$\text{Or,} \qquad 1800x = 168{,}000$$

$$\therefore \; x = \frac{168{,}000}{1800} = 93\tfrac{1}{3} \text{ days.}$$

(16.) A cask can be filled by each of 3 taps in 12, 15, and 18 minutes respectively: find in what time it could be filled by all three taps running together.

(17.) Two coaches start at the same time from York and London, a distance of 200 miles; the one from London travels at $9\frac{1}{2}$ miles an hour, and that from York at $8\frac{3}{4}$: where and when will they meet?

(18.) A merchant lays by at the end of each year one-third of his capital, except an annual expenditure of £500; at the end of 7 years he is worth £10,000: what was his original capital?

(19.) A person sets out on a journey at the rate of 30 miles a day, but falls off in his speed 4 miles daily; four days afterwards another sets out from the same place on the same route, travelling 50 miles the first day, but falling off 4 miles daily: find when one will overtake the other.

(20.) If $A$ and $B$ have £600, $B$ and $C$ £800, and $A$ and $C$ £1000: find the sums possessed by $A$, $B$, $C$, separately.

(21.) A general sent $\frac{1}{7}$ of his army to take possession of a fortress, and 756 men more to attack a detachment of the enemy; and he then found that only $\frac{2}{3}$ of his men were left: find the number of his army.

(22.) If $n$ shillings be distributed among $a$ persons, some receiving 10*d.* each and the rest 14*d.* each: find how many receive 10*d.* and how many receive 14*d.*

(23.) If 11*s.* be distributed among 12 persons, as in the last question; find the number of those who receive 10*d.* and 14*d.* respectively.

(24.) A son's age is one-half of his father's, but 20 years ago it was only one-sixth of his father's: find the age of each.

(25.) Four reapers begin to cut a field of corn; on the first day they cut one-fourth of the field, on the next day they cut one-fifth of what is left, and half an acre more; and then only half the field is cut: find the number of acres in the whole field.

(26.) There are 20 coins in my purse, and they consist of half-sovereigns and half-crowns: the value of the whole is £7 15*s.*: find how many there are of each sort.

(27.) 124 apples and oranges cost 3*s.*; the apples are sold at 5 a penny, and the oranges at 3 for 2*d.*: find the number of each kind of fruit.

(28.) A cask contains a mixture of spirit and water, 106 gallons in all; now, if 8 gallons of water be added to the mixture it is found that the water in the cask is one-fourth of the whole: find the quantities of spirit and of water in the cask at first.

(29.) Find at what time between 1 and 2 the minute and hour hands of the clock are together.

Let $x$ = the number of minute divisions between the figure 1 and the point at which the hands coincide.

Then $x$ = the number of divisions passed over by the hour hand since 1 o'clock.

And $5+x$ = the number of divisions passed over by the minute hand since 1 o'clock.

Now the number of divisions passed over by the minute hand is twelve times the number passed over by the hour hand in the same time.

$$\therefore \quad 5+x = 12x$$

$$\text{Or,} \quad 11x = 5$$

$$\therefore \quad x = \tfrac{5}{11} \text{ of a minute} = 27\tfrac{3}{11}''.$$

That is, the time required is $5'\ 27''\ \tfrac{3}{11}$ after 1.

In like manner the following cases may be solved:—

1. At what time between 9 and 10 will the hands be together?
2. At what time between 9 and 10 will they be perpendicular to each other?
3. At what times between 10 and 11 are they perpendicular to each other.
4. At what time between 6 and 7 are they exactly opposite to each other?

(30.) The difference of the squares of two consecutive numbers is 381: what are the numbers?

(31.) A farmer has oxen worth £12 10*s.* each, and sheep worth £1 18*s.* each; the whole number of oxen and sheep together is 35, and the whole value is £426 18*s.*: find the number of oxen and sheep respectively?

(32.) An estate of 450 acres of land is to be divided equally between two brothers; the first is to have 180 acres of land, and the second the rest, and the first is to pay the second 9000 guineas: find the value of the land per acre.

(33.) A house and garden cost 800 guineas, and the price of the house is to the price of the garden as 7 : 3: find the price of each.

(34.) Two persons set out at the same time from Plymouth and

Exeter to meet each other; their rates of walking are 3 and 4 miles an hour respectively; they pass each other and reach their destinations, and, after remaining two hours in Exeter and Plymouth respectively, they return home again: find the place of their first and second meeting; the distance between the towns being 49 miles.

(35.) A gentleman's income, after paying poor-rates, and an income-tax of 7*d.* in the pound, is £486; the poor-rate is £22 10*s.* more than the income-tax: find the original income, and the amount in the pound of the poor-rate.

(36.) One cask contains 12 gallons of wine and 18 of water; and another cask contains 9 gallons of wine and 3 of water: how many gallons must be drawn from each cask so as to make a mixture of 7 gallons of wine and 7 of water?

(37.) Two men set out on a journey, walking at the rate of 4 miles an hour; after walking for six hours, one of the two lessens his rate to 3 miles an hour; the other continues on at the same rate, and arrives at the end of his journey an hour before his companion: find the length of their journey.

(38.) A rower who can pull at the rate of 6 miles an hour, can pull 10 miles down the river in half the time that he can pull 10 miles up it: find the rate at which the river flows.

(39.) The duty on tea was formerly 2*s.* per lb., but in consequence of a reduction in the duty, the quantity of tea consumed was increased by one-fourth, and the amount of revenue was increased by one twenty-fourth: find the reduction in the duty per lb.

(40.) The sum of three numbers is 74; and if the second be divided by the first, or the third be divided by the second, the quotient is 2, and the remainder 1: find the three numbers.

(41.) A railroad runs from $A$ to $C$; a goods train starts from $A$ at 12 o'clock, and a passenger train at 1 o'clock; after going two-thirds of the distance the goods' train breaks down, and can only travel at three-fourths of its former speed. At 20 minutes before 3 a collision occurs, 10 miles from $C$. The rate of the passenger train being double of the diminished speed of the goods' train: find the distance from $A$ to $C$, and the rates of the two trains in miles per hour.

(42.) A luggage train going at the rate of 10 miles an hour is some distance in advance of an express engine, which starts to overtake it, and just comes up with it in 5 hours; had the express engine travelled 10 miles an hour less, it would have been $7\frac{1}{2}$ hours in coming up with the luggage train: find the distance between the train and the express engine at first, and the rate at which the latter travelled.

### 70. *Simple Equations of two or more Unknown Quantities.*

In the equations of this chapter we have hitherto supposed only one unknown quantity, $x$. We may however have equations with two or more unknown quantities in them.

Thus $$3x-y = 13$$

is an equation of two unknowns, $x$ and $y$.

When one equation contains two unknowns, the values of $x$ and $y$ cannot be determined from it alone. In fact, an infinite number of values can be found to satisfy such an equation. For example, the following are values of $x$ and $y$ that satisfy the above equation.

$$\left.\begin{matrix}x=5\\y=2\end{matrix}\right\}\quad\left.\begin{matrix}x=6\\y=5\end{matrix}\right\}\quad\left.\begin{matrix}x=7\\y=8\end{matrix}\right\}\quad\left.\begin{matrix}x=8\\y=11\end{matrix}\right\}\quad\left.\begin{matrix}x=9\\y=14\end{matrix}\right\}$$

If, however, there be *two* equations in $x$ and $y$, it will be found that one value and only one value of each of those letters can be found to satisfy the two equations.

Thus if we have the two equations

$$\begin{aligned}3x-y &= 13 \quad (1.)\\ x+y &= 7 \quad (2.)\end{aligned}$$

Adding (1.) and (2.) together, we find

$$4x = 13+7$$

Or, $$4x = 20$$

$$\therefore \quad x = 5$$

And $$y = 7-x=2.$$

Therefore 5 and 2 are the values of $x$ and $y$, which satisfy both the equations, and no other values could be found to satisfy them.

Equations (1.) and (2.) are called *simultaneous* equations, because they are *simultaneously* satisfied by the same values of $x$ and $y$.

71. The general form of simultaneous equations is the following:—

$$\begin{aligned}ax+by &= c \quad (1.)\\ dx+ey &= f \quad (2.)\end{aligned}$$

It will now be shown generally that there is one pair and only one pair of values of $x$ and $y$ which satisfies the two equations.

First, multiply (1.) by $e$ and (2.) by $b$: we have

$$acx+bey = ce$$

And $$\underline{bdx+bey} = \underline{bf}$$

Subtract $\therefore$ $acx-bdx = ce-bf$

Or, $x(ae-bd) = ce-bf$

$$\therefore \quad x = \frac{ce-bf}{ae-bd}$$

Again, multiply (1.) by $d$ and (2.) by $a$, we have

$$adx+bdy = cd$$
$$adx+aey = af$$

Subtract $\therefore$ $bdy-acy = cd-af$

Or, $y(ae-bd) = af-cd$

$$\therefore \quad y = \frac{af-cd}{ac-bd}.$$

Thus one value for $x$ and one for $y$ has been obtained, which are simultaneous, and which satisfy the equations.

But besides these, no others can be found to satisfy the equation; for, calling the values of $x$ and $y$ just found $\alpha$ and $\beta$ respectively, let, if possible, $\alpha'$ $\beta'$ be two other values of $x$ and $y$ which satisfy it.

Then we have $a\alpha+b\beta = c$

$$\underline{a\alpha'+b\beta'} = \underline{c}$$

By subtraction $a(\alpha-\alpha') + b(\beta-\beta') = 0$

Similarly, $c(\alpha-\alpha') + d[\beta-\beta') = 0.$

Multiply the first of these by $d$, and the second by $b$, then

$$ad(\alpha-\alpha') + bd(\beta-\beta') = 0$$
$$bc(\alpha-\alpha') + bd(\beta-\beta') = 0.$$

By subtraction $(ad-bc)(\alpha-\alpha') = 0.$

Now one of these two factors must $= 0$; but $ad-bc$ does not equal zero necessarily, therefore $\alpha-\alpha' = 0$ or $\alpha=\alpha'$; that is, the second value of $x$ is the same as the first, or there are not two different values of $x$ which can satisfy the equations.

In the same way it can be shown that there are not two different values of $y$ which satisfy the equations.

## Examples.

Find the values of $x$ and $y$ which satisfy the two equations.

$$3x+5y = 37 \quad (1.)$$
$$5x+2y = 30 \quad (2.)$$

Multiply both sides of (1.) by 2, and both sides of (2.) by 5.

They become $6x+10y = 74$

And $\underline{25x+10y} = \underline{150}.$

Subtracting the lower from the upper, we get

$$-19x = -76$$
$$\therefore \quad x = 4$$

And from (1.) $3\times4+5y = 37$

$\therefore$ $5y = 37-12$

Or, $5y = 25$

$\therefore$ $y = 5$

**72.** The following rule may be given for the solution of two simultaneous equations containing two unknown quantities.

(a.) *Clear both equations of fractions.*

(b.) *Transpose all the terms containing* x *or* y *to the first side of the equation; and all other terms to the second side; and simplify by collecting the coefficients of* x *and* y.

(c.) *Multiply one or both of the equations by such numbers as will make the coefficients of* y *equal in the two equations.*

(d.) *If these coefficients be of the same sign, subtract one equation from the other; the remaining equation will contain only one unknown,* x: *this may immediately be solved as a simple equation of one unknown,* viz., x. *But if the equal coefficients be of different signs, add the equations together.*

(e.) *When* x *is known,* y *may be found by substituting for* x *its value in one of the original equations.*

### Examples.

$$x-4 = \tfrac{1}{7}(y+5) \qquad (1.)$$
$$2y-\tfrac{3}{2} = \tfrac{1}{6}(x+10) \qquad (2.)$$

By $(a)$ (1) becomes $7x-28 = y+5$.

By $(b)$ we get $7x-y = 33$.

Similarly, (2) becomes $12y-9 = x+10$

Or, $12y-x = 19$

But $7x-y = 33$.

By $(c)$ the two equations become

$$\begin{aligned} 12y-x &= 19 \\ \underline{-12y+84x} &= \underline{396} \end{aligned}$$

By $(d)$ $83x = 415$

$\therefore$ $x = \tfrac{415}{83}$

Or, $x = 5$

To get $y$, substitute this value of $x$ in the equation

$$12y - x = 19$$

This gives $12y - 5 = 19$

Or, $12y = 24$

$\therefore$ $y = \frac{24}{12}$

Or, $y = 2$

$\therefore$ $\left.\begin{aligned} x &= 5 \\ y &= 2 \end{aligned}\right\}$ are the values required.

*Note.*—Many equations occur in the following pages, which can be more easily solved by methods somewhat different from that just given. These methods will best be learnt by practice, and by carefully noticing the solutions as they occur.

### Examples XXXI.

(1.) $$\left.\begin{aligned} 3(x+y) &= 9+y \\ 10-4y &= x-y \end{aligned}\right\}$$

(2.) $$\left.\begin{aligned} 9x-4y &= 8 \\ 12x-3y &= 27 \end{aligned}\right\}$$

(3.) $$\left.\begin{aligned} 3x-2y &= 96 \\ 8x-5y &= 480 \end{aligned}\right\}$$

(4.) $$\left.\begin{aligned} 2(x-4y) &= 11+y \\ 4(x-3y) &= 15+x \end{aligned}\right\}$$

(5.) $$\left.\begin{aligned} \frac{x}{5}+3y+1 &= \frac{58}{5} \\ \frac{y}{9}+\frac{x}{3} &= 3 \end{aligned}\right\}$$

(6.) $$\left.\begin{aligned} x+y-12 &= \frac{3-y}{4} \\ x-2y &= \frac{3}{8}(3-y) \end{aligned}\right\}$$

(7.) $$\left.\begin{aligned} \frac{3x}{7} &= 1+y \\ 11(x-8)+1+5y &= 0 \end{aligned}\right\}$$

(8.) $$\left.\begin{aligned} 3x+4y &= 34 \\ y+\frac{2x}{3} &= 8 \end{aligned}\right\}$$

(9.) $$\left.\begin{aligned} x+y &= a \\ x-y &= b \end{aligned}\right\}$$

To solve these, add the two equations together: we find

$$2x = a+b$$

$$\therefore \quad x = \frac{a+b}{2}$$

Again, to find $y$, subtract the lower from the upper.

Then $2y = a-b$

Or, $y = \frac{a-b}{2}$

Now $a$ and $b$ are *known* quantities, and the equations are therefore solved when $x$ and $y$ are expressed in terms of $a$ and $b$. Similar results may always be expected when *letters*, and not *numbers*, represent the known quantities.

(10.) $$\left.\begin{aligned} \frac{x}{2}+3y &= a \\ \frac{x}{2}-3y &= b \end{aligned}\right\}$$

(11.) $$\left.\begin{aligned} \frac{x}{3}+3y &= 21 \\ \frac{y}{3}+3x &= 29 \end{aligned}\right\}$$

(12.) $$\left.\begin{aligned}\frac{x}{3}-\frac{3y}{10}&=\frac{11}{5}\\ \frac{3x}{10}-\frac{y}{3}&=2\tfrac{7}{30}\end{aligned}\right\}$$ (13.) $$\left.\begin{aligned}2\,(x+y)&=24-x\\ 2\,(x+y)&=21-y\end{aligned}\right\}$$

To solve (13) proceed thus :—

$$24-x = 21-y$$

Or, $$y-x = 21-24$$

$\therefore$ $$y = x-3$$

Substitute this value of $y$ in the first of the two given equations.

Then $$2\,(x+x-3) = 24-x$$

Or, $$4x-6 = 24-x$$

$\therefore$ $$5x = 30$$

Or, $$x = 6$$

And, $$y = x-3$$

Or, $$y = 3$$

This method is shorter than the general method given in 72, and should be used in all similar cases.

(14.) $$\left.\begin{aligned}\frac{x+4}{y}&=2\\ \frac{x}{y+1}&=\tfrac{1}{2}\end{aligned}\right\}$$ (15.) $$\left.\begin{aligned}\frac{x}{4}+\frac{y}{5}&=10-\tfrac{2}{3}\,(x+y)\\ \frac{y}{12}&=7-\frac{3x-y}{3}\end{aligned}\right\}$$

(16.) $$\left.\begin{aligned}\frac{y}{3}-\frac{x}{2}+16&=0\\ \frac{y}{8}-\frac{x}{5}+12&=0\end{aligned}\right\}$$ (17.) $$\left.\begin{aligned}\frac{y-2}{4}&=\frac{x+3}{5}-\frac{7-x}{3}\\ \frac{x-3}{8}&-\frac{2y-8}{3}-\frac{2x+y}{16}\end{aligned}\right\}$$

(18.) $$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{c}{x}+\frac{d}{y}&=n\end{aligned}\right\}$$ (19.) $$\left.\begin{aligned}\frac{m}{x}+\frac{n}{y}&=a\\ \frac{n}{x}+\frac{m}{y}&=b\end{aligned}\right\}$$

*Note.*—Solve these like (13) after clearing of fractions.

(20.) $$\left.\begin{aligned}\frac{a+x}{b+y}&=\frac{b}{a}\\ ax+by&=c\end{aligned}\right\}$$ (21.) $$\left.\begin{aligned}(x+5)(y-1)&=(x-1)(2y-2)\\ x+y&=1\end{aligned}\right\}$$

(22.) $$\left.\begin{aligned}\frac{bx+ay}{xy}&=\frac{a}{c}\\ \frac{ax-by}{xy}&=\frac{b}{c}\end{aligned}\right\}$$ (23.) $$\left.\begin{aligned}4(x+y)+3(x-y)&=708\\ 3(x+y)-4(x-y)&=6\end{aligned}\right\}$$

(24.) $$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1}{a}+\frac{1}{b}\\ \frac{2}{x}+\frac{3}{y}&=\frac{2}{a+b}\end{aligned}\right\}$$ (25.) $$\left.\begin{aligned}a(x+y-a)&=b(x-y-b)\\ (a^2-b^2)(x-y)&=ab(a-b)\end{aligned}\right\}$$

(26.) $$\left.\begin{aligned}(x+b)(y-c)+a &= (x-b)(y+c)-a\\ (x+c)(y-b) &= (x+b)(y-c)\end{aligned}\right\}$$

(27.) $$\frac{x}{a}+\frac{y}{b}=\frac{x-a}{b}+\frac{y-b}{a}=1$$

(28.) $$\left.\begin{aligned}\frac{7x-5}{11}+\frac{4y-6}{3} &= x-\frac{9-5x}{2}+\frac{x-3y}{5}-2\\ \frac{3}{5}(x+3 &= \frac{2}{3}y\end{aligned}\right\}$$

(29.) $$\left.\begin{aligned}ax+by &= c\\ (a+a')x+(b+b')y &= c+c'\end{aligned}\right\}$$

(30.) $$\left.\begin{aligned}x:y&::5:7\\ \frac{x}{4}+3y &= 39\end{aligned}\right\}$$

(31.) $$\left.\begin{aligned}2x-\frac{y+3}{4} &= 7+\frac{3y-2x}{5}\\ 4y+\frac{x-2}{3} &= 26\tfrac{1}{2}-\frac{2y+1}{2}\end{aligned}\right\}$$

(32.) $$\left.\begin{aligned}5\left(\frac{x}{2}-1\right) &= \tfrac{1}{2}+\tfrac{3}{2}(y+1)\\ \tfrac{3}{4}x-\tfrac{1}{2}(y-5) &= 11x-\frac{30x-44}{3}\end{aligned}\right\}$$

**73.** Equations sometimes occur of three or more unknown quantities; and it can be proved by similar reasoning to that used in 71 that the number of equations must be the same as the number of unknowns. For instance, if there be *three* unknowns, $x$, $y$, $z$, there must be given *three* equations.

The mode of proceeding in such cases will be shown in the following example:—

Let $$x+y+z = 12 \quad (1.)$$
$$3x-2y+2z = 10 \quad (2.)$$
$$4x+3y-z = 14 \quad (3.)$$
be the three given equations.

Multiply (1.) by 2, and subtract the result from (2.)

Then $$x-4y = 14 \quad (4.)$$

Again, add (1.) to (3.),

This gives $$5x+4y = 26 \quad (5.)$$

Now (4.) and (5.) are two equations, containing only *two* unknowns, and from these the values of $x$ and $y$ can be found as before shown.

Hence we get $$x = 2$$
$$y = 4$$

To find $z$, return to (1.): substitute in it the values of $x$ and $y$: it then becomes

$$2+4+z = 12$$
$$\therefore \quad z = 6$$

Hence to solve three equations containing three unknowns, $x$, $y$, $z$, observe the following

### RULE.

(a.) *Take any* two *of the three given equations and make the coefficients of* z *equal in these; then by adding or subtracting obtain an equation in* x *and* y *only.*

(b.) *Take the remaining equation and either of the others, and from these two obtain in like manner an equation in* x *and* y *only.*

(c.) *Solve the two equations just found in* x *and* y *by the methods already given.*

(d.) *Substitute the values of* x *and* y *in one of the original equations; then the value of* z *will be found.*

### Examples XXXII.

(1.)
$$\left.\begin{aligned} 3x+2y+3z &= 40 \\ x-2y+4z &= 21 \\ 12x+4y-6z &= 14 \end{aligned}\right\}$$

(2.)
$$\left.\begin{aligned} 4x-3y+z &= 37 \\ 2x+3y+4z &= 38 \\ x-y-z &= 5 \end{aligned}\right\}$$

(3.)
$$\left.\begin{aligned} x+y-z &= 0 \\ 4x+5y+6z &= 32 \\ 4x+3y-8z &= -14 \end{aligned}\right\}$$

(4.)
$$\left.\begin{aligned} x-y+z &= 3 \\ x-2y+3z &= 8 \\ x-3y+6z &= 17 \end{aligned}\right\}$$

(5.)
$$\left.\begin{aligned} z+y-x &= 2 \\ 6x-3y-8z &= -2 \\ 3x-y-4z &= 2 \end{aligned}\right\}$$

(6.)
$$\left.\begin{aligned} x+y &= 1 \\ y+z &= 2 \\ x+z &= 3 \end{aligned}\right\}$$

(7.)
$$\left.\begin{aligned} \frac{x}{4}-\frac{y}{2}+z &= 42 \\ 3x+y-\tfrac{3}{2}z &= 28 \\ \tfrac{3}{2}(x+z)+y &= 160 \end{aligned}\right\}$$

(8.)
$$\left.\begin{aligned} (x-a)+(y-b)+(z-c) &= 0 \\ (b-c)x+(c-a)y+(a-b)z &= 0 \\ c(x-c)+a(y-a)+b(z-b) &= 0 \end{aligned}\right\}$$

(9.)
$$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{5}{6} \\ \frac{1}{y}+\frac{1}{z} &= \frac{7}{12} \\ \frac{1}{x}+\frac{1}{z} &= \frac{3}{4} \end{aligned}\right\}$$

(10.)
$$\left.\begin{aligned} mx+ny+pz &= m \\ m^2x+n^2y+p^2z &= m^2 \\ m^3x+n^3y+p^3z &= m^3 \end{aligned}\right\}$$

(11.)
$$\left.\begin{aligned} \frac{x}{2}+\frac{y}{3}+\frac{z}{4} &= 62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5} &= 47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6} &= 38 \end{aligned}\right\}$$

(12.)
$$\left.\begin{aligned} 6x+ay &= c \\ cy+bz &= a \\ az+cx &= b \end{aligned}\right\}$$

(13.)

$$\left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z} &= 76 \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z} &= 161 \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z} &= \frac{305}{3} \end{aligned}\right\}$$

*Note.*—(13.) is most easily solved by putting

$$\frac{1}{x} = x^1\,;\ \frac{1}{y} = y^1\,;\ \frac{1}{z} = z^1$$

The first equation then becomes

$$3x^1-\frac{4}{5}y^1+z^1 = 76$$

and the others are changed in like manner; $x^1$, $y^1$, $z^1$, can then be found, and the values of $x, y, z$ are then known,

Since

$$x = \frac{1}{x^1};\ y = \frac{1}{y^1};\ z = \frac{1}{z^1}$$

(14.)

$$\left.\begin{aligned} x+y+z &= 0 \\ (a+b)x+(a+c)y+(b+c)z &= 0 \\ abx+acy+bcz &= 1 \end{aligned}\right\}$$

(15.)

$$\left.\begin{aligned} \frac{105}{xyz} &= 1 \\ \frac{35}{yz} &= \frac{3}{x} \\ \frac{7}{z} &= \frac{15}{xy} \end{aligned}\right\}$$

(16.)

$$\left.\begin{aligned} xyz &= a(yz-zx-xy) \\ xyz &= b(xz-xy-yz) \\ xyz &= c(xy-yz-xz) \end{aligned}\right\}$$

(17.)

$$\left.\begin{aligned} 9xy &= 20(x+y) \\ 10xz &= 24(x+z) \\ 11yz &= 30(y+z) \end{aligned}\right\}$$

(18.)

$$\left.\begin{aligned} xz &= 3(x+z) \\ xy &= 8(x+y) \\ 7yz &= 9(y+z) \end{aligned}\right\}$$

(19.)

$$\left.\begin{aligned} \frac{14y+7z}{29+7yz} &= \frac{7}{y+3z} \\ \frac{x-z}{2xz+9} &= \frac{1}{x+3z} \\ \frac{5}{2y+3x} &= \frac{4x-y}{xy+86} \end{aligned}\right\}$$

(20.)

$$\left.\begin{aligned} x &= 2ay-4a^2z+8a^3 \\ x &= 2by-4b^2z+8b^3 \\ x &= 2cy-4b^2z+8b^3 \end{aligned}\right\}$$

(21.)

### 74. *Problems involving Equations of two or more Unknown Quantities.*

### Examples XXXIII.

(1.) There are two numbers such, that six times the greater is equal to seven times the less, and if one-third of the less be taken from one-half of the greater, the remainder will be 3: find the numbers.

*Solution.* Let $x$ = the greater number,

$y$ = the less.

$$\left.\begin{aligned} \text{Then} \quad & 6x = 7y \\ \text{And} \quad & \frac{x}{2} - \frac{y}{3} = 3 \end{aligned}\right\} \qquad \begin{aligned} &(1.) \\ &(2.) \end{aligned}$$

are the two equations to be solved.

Hence 14 and 12 are the two numbers.

Since $x = 14$

And $y = 12$

(2.) Find a fraction such, that if its numerator and denominator be both increased by 5, the value is $\frac{1}{2}$, and if they be both diminished by 1, the value is $\frac{1}{4}$.

*Solution.* Let $x$ = the numerator,

$y$ = the denominator.

$$\text{Then} \quad \left.\begin{aligned} \frac{x+5}{y+5} &= \tfrac{1}{2} \\ \frac{x-1}{y-1} &= \tfrac{1}{4} \end{aligned}\right\}$$

Are the two equations to be solved.

They give $x = 4;\ y = 13$,

$\therefore$ $\frac{4}{13}$ is the required fraction.

(3.) If the greater of two numbers be multiplied by 2 and the less by 3, the sum of the two products will be 20; but if the greater be multiplied by 3 and the less by 2, the difference of the products will be 17: find the two numbers.

(4.) What fraction is that to the numerator of which if 4 be added, the value is $\frac{1}{2}$, and if 7 be added to the denominator, the value is $\frac{1}{5}$?

(5.) A bill of £26 5*s*. was paid with guineas and crowns, and the number of crowns was 12 more than twice the number of guineas: find the number of each coin.

(6.) Divide the number 216 into 3 parts such that $\frac{1}{5}$ of the first part $= \frac{1}{6}$ of the second part $= \frac{1}{7}$ of the third part.

*Note.*—Solve this 1st with two unknowns.
2nd with three.

(7.) The sum due to two labourers for wages is £1 9*s*., the first having worked 6 days, the other 8 days; and the first earns in 3 days 6*d*. more than the second does in 4 days: find the daily wages of each.

(8.) My purse contains sovereigns and half crowns only; and the gold is worth six times as much as the silver. After paying a debt of £4 10*s*. out of it with an equal number of sovereigns and half-crowns, I find that the gold is worth 4 times as much as the silver: find the *number* of sovereigns in the purse at first.

(9.) A market woman bought apples at 5 a penny, and oranges at 2 a penny; she then sold the apples at the rate of 3 for 1*d.*, and the oranges at the rate of 3 for 2*d.*, and gained 1*s.* by the sale: find the number of each kind of fruit.

(10.) There is a number of two digits; and it is found that if both the digits be diminished by 4, the number thus formed is one less than the half of the original number; also the left-hand digit of the original number is twice as great as that of the reduced number: find the number.

*Solution.* Let $x$ = the left-hand digit of the original number.
$y$ = the right-hand one.
Then $x-4$ = the left-hand digit of the reduced number,
And $y-4$ = the right-hand one.
Also $10x+y$ = the original number.
And $10(x-4)+(y-4)$ = the reduced number.
∴ By question

$$\frac{10x+y}{2} = 10(x-4)+(y-4)+1 \qquad (1.)$$

And

$$x = 2(x-4) \qquad (2.)$$

Are the equations to be solved:
They give $x = 8$
$y = 6$
∴ 86 is the number.

*Note.*—This mode of representing a number when its digits are given should be observed.

(11.) If 18 be added to a certain number of two digits, they become inverted, and the latter number is 1¾ times as great as the original number: find the original number.

(12.) A draper bought two kinds of cloth at 7*s.* 6*d.* and 10*s.* 6*d.* a yard respectively, and the cost of the whole was £19 10*s.*; he sells the cloth again at 8*s.* and 12*s.* a yard respectively, and gains on the whole £2 10*s.*: find the quantity of each kind of cloth.

(13.) *A* and *B* working together could build a wall in 90 days, but one of them falls ill and is obliged to leave off working for 10 days, and the work is not completed till the end of 96 days: find the time in which *A* alone could do the whole.

*Solution.* Let $x$ = the number of days in which *A* alone could build the wall,
And $y$ = the number of days in which *B* alone could build the wall.
Let $W$ = the whole work to be done in rods.

Then $\frac{W}{x}$ = the quantity of work done by $A$ alone in one day,

and $\frac{W}{y}$ = the quantity done by $B$ alone in one day.

$\therefore$ $96\frac{W}{x} + 86\frac{W}{y}$ = the whole work done by $A$ and $B$.

Hence $96\frac{W}{x} + 86\frac{W}{y} = W$ (1.)

Also $90\frac{W}{x} + 90\frac{W}{y} = W$ (2.) from the question.

Dividing both sides of both equations by $W$, we find

$$\frac{96}{x} + \frac{86}{y} = 1$$

And

$$\frac{90}{x} + \frac{90}{y} = 1$$

From these equations we find

$x = 225$, the number of days in which $A$ could do the work by himself.

Similarly $y$ may be found.

*Note.*—The quantity of work is assumed to be $W$ for convenience sake, in order that the statements of the problem may be put into algebraical language; but $W$ is not required to be known in any other terms: in fact, it is clear that the result would have been the same, if $W$ had represented any other quantity or kind of work than it does.

(14.) If $A$ do a piece of work in 20 days which $A$ and $B$ can do together in 15 days, how long would $B$ take to do it alone?

(15.) A cistern of 240 gallons can be filled by two pipes running together in 40 minutes; but if one pipe be stopped at the end of half an hour, the whole time of filling will be 55 minutes: find the number of gallons per minute conveyed by each pipe.

(16.) Two persons each receive a legacy of £300, and one of them then has three times as much money as the other; but if they had each received a legacy of £800, the richer would have had only twice as much money as the other: find the sum each had originally.

(17.) In going a mile, the fore-wheel of a carriage turns round 264 times more than the hind-wheel; but if the circumference of each wheel were one yard more than it is, the fore would make only 176 more revolutions than the hind-wheel in going a mile; find the circumference of each wheel.

(18.) Three persons were engaged to do a piece of work, which would have lasted them 18 hours if they had all begun at the same time; but instead of doing so, they began at equal intervals, and then continued to work till the whole was finished; the first received twice as much as the last: find the time occupied, each man's wages being proportioned to the time that he worked.

(19.) The rent-charge of a farm consists of a certain number of quarters of wheat and of rye; now, when the wheat is at 56*s.* the quarter and rye at 24*s.*, the values of the wheat and the rye are the same; but when the wheat is at 60*s.* and the rye at 25*s.*, the rent is increased by £9: find the amount of the rent-charge.

(20.) A person has 80 gallons of wine in a cask, and wishes to draw off half of it, but not having a 40-gallon cask at hand, he does it by means of two other measures, *A* and *B*, which together hold 80 gallons, in the following way:—

1st. He fills the larger measure (*A*) from the cask.

2nd. He fills the smaller measure (*B*) from the larger (*A*), and empties (*B*) back into the cask, and then pours what is left in (*A*) into (*B*).

3rd. He fills (*A*) from the cask, and then pours from (*A*) into (*B*) till (*B*) is full; half of the whole quantity of wine will then be found in (*A*): find the capacity of the two measures.

(21.) Find a quantity such, that whether it be divided into three or four equal parts, the continued product of the parts shall be the same.

(22.) A farmer sells to one person 6 horses and 3 cows for £180, and to another he sells 4 horses and 4 cows for £144: find the price of a horse and of a cow.

(23.) If from half the sum of the three sides of a triangle, each side be subtracted separately, the three remainders are $6\frac{1}{2}$, $5\frac{1}{2}$, and $4\frac{1}{2}$ yards respectively: find the length of each side of the triangle.

(24.) A bill of 25 guineas was paid with crowns and half-guineas, and twice the number of half-guineas exceeded three times the number of crowns by 17; how many were there of each coin?

(25.) Two labourers, *A* and *B*, dig for two hours, after which *A* goes away, and *B* digs the remaining part of the work in 2h. 48m.; but if *B* had left at the end of the two hours, and *A* had continued digging, it would have taken him 4h. 40m. to finish the work: find in what time each would have done it separately.

(26.) A general finds that his cavalry with half his artillery and infantry together, or his artillery with one-third of his cavalry and infantry together, or his infantry with one-fourth of his cavalry and artillery together, make up the same number of men, *viz.* 5950: how many men were there in each arm of the service?

# CHAPTER X.

## ON THE RESOLUTION OF ALGEBRAICAL EXPRESSIONS INTO FACTORS.

**75.** In the articles of this chapter we shall discuss some of the most useful methods of resolving compound expressions into their factors.

### BINOMIALS.

By Art. (50) it appears that any Binomial which expresses the difference of the squares of two quantities can always be resolved into two factors. Some examples will here be given in this formula of a more complicated kind than those in Chap. VIII. The method of dealing with such expressions will be seen from the examples worked out below.

$$\begin{aligned}
\text{Ex. 1.}\qquad a^8-x^8 &= (a^4)^2-(x^4)^2\\
&= (a^4+x^4)(a^4-x^4)\\
\text{But}\qquad a^4-x^4 &= (a^2+x^2)(a^2-x^2)\\
&= (a^2+x^2)(a+x)(a-x)\\
\therefore\qquad a^8-x^8 &= (a^4+x^4)(a^2+x^2)(a+x)(a-x)
\end{aligned}$$

In this example the principle has been applied three times in succession; first to $a^8-x^8$, then to $a^4-x^4$, then to $a^2-x^2$.

$$\begin{aligned}
\text{Ex. 2.}\qquad \tfrac{9}{16}x^2y^2-z^4 &= (\tfrac{3}{4}xy)^2-(z^2)^2\\
&= (\tfrac{3}{4}xy+z^2)(\tfrac{3}{4}xy-z^2)\\
\text{Ex. 3.}\qquad (3x-1)^2-4 &= \{(3x-1)+2\}\{(3x-1)-2\}\\
&= (3x+1)(3x-3)\\
&= 3(3x+1)(x-1)
\end{aligned}$$

Similarly, the following may be resolved:

### Examples XXXIV.

$\dfrac{x^2}{4}-\dfrac{y^2}{16}$; $\dfrac{4a^2}{6^2}-\dfrac{6^2}{4a^2}$; $16a^4-81b^4$; $256x^4y^4z^4-1$; $\dfrac{x^2}{9}-\dfrac{9}{x^2}$;

$\left(x+\dfrac{1}{x}\right)^2-1$; $(a+b)^4-a^2b^2$; $(x+1)^2-4$; $(x^4-1)^2-(x^2-1)^2$;

$$\frac{1}{16x^4}-\frac{x^4}{81};\ (a+b+c)^2-a^2;\ (x^2y^2-y^2z^2)^2-x^4z^4;$$

$$\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^4-\left(\frac{x^2}{z}+\frac{z^2}{x}\right)^2;\ \left(\frac{x+2y}{x+y}\right)^2-\left(\frac{x}{y}\right)^2;$$

$$\frac{16}{25}\left(\frac{a+x}{a-x}\right)^2-\frac{25}{16}\left(\frac{a-x}{a+x}\right)^2;\quad a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2;$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2-\frac{4x^2y^2}{a^2b^2}.$$

**76.** Since

$$\begin{aligned} a^3-b^3 &= (a-b)\ (a^2+ab+b^2) \\ \text{and}\quad (a^3+b^3) &= (a+b)\ (a^2-ab+b^2), \end{aligned}$$

it follows that any Binomial which consists of the difference of the cubes of two quantities, or of the sum of the cubes of two quantities, can always be resolved into two factors; and these will be of the forms given above.

Ex. 1.
$$\begin{aligned} 8x^3y^3-1 &= (2xy)^3-1^3 \\ &= (2xy-1)\{(2xy)^2+(2xy).1+1^2\} \\ &= (2xy-1)\ (4x^2y^2+2xy+1) \end{aligned}$$

Ex. 2.
$$\begin{aligned} \tfrac{8}{27}a^3d^3+\tfrac{27}{8}b^3c^3 &= (\tfrac{2}{3}ad)^3+(\tfrac{3}{2}bc)^3 \\ &= (\tfrac{2}{3}ad+\tfrac{3}{2}bc).\{(\tfrac{2}{3}ad)^2-\tfrac{2}{3}ad.\tfrac{3}{2}bc+(\tfrac{3}{2}bc)^2 \\ &= (\tfrac{2}{3}ad+\tfrac{3}{2}bc).(\tfrac{4}{9}a^2d^2-abcd+\tfrac{9}{4}b^2c^2) \end{aligned}$$

Ex. 3.
$$\begin{aligned} a^9+x^9 &= (a^3)^3+(x^3)^3 \\ &= (a^3+x^3)\{(a^3)^2-a^3x^3+(x^3)^2\} \\ &= (a+x)\ (a^2-ax+x^2)\ (a^6-a^3x^3+x^6) \end{aligned}$$

Ex. 4.
$$\begin{aligned} \frac{a^6}{64}-\frac{64b^6}{729} &= \left(\frac{a}{2}\right)^6-\left(\frac{2b}{3}\right)^6 \\ &= \left\{\left(\frac{a}{2}\right)^3+\left(\frac{2b}{3}\right)^3\right\}\left\{\left(\frac{a}{2}\right)^3-\left(\frac{2b}{3}\right)^3\right\} \\ &= \left(\frac{a}{2}+\frac{2b}{3}\right)\left(\frac{a^2}{4}-\frac{ab}{3}+\frac{4b^2}{9}\right)\left(\frac{a}{2}-\frac{2b}{3}\right)\left(\frac{a^2}{4}+\frac{ab}{3}+\frac{4b^2}{9}\right) \end{aligned}$$

In this example, the two methods of Arts. 50 and 76 have been used.

### Examples XXXV.

Resolve the following into all their factors:—

$$x^6-64y^6;\ 8a^6+27b^6;\ 125x^3+1;\ 64a^6-b^6;\ 64a^6b^3+27c^3d^6;$$

$$\frac{a^6b^6}{64}-64c^{12};\ 56ab^4c-7abc^4;\ 64a^{12}+729b^{12};\ \frac{1}{27}a^4b^4cd+\frac{125}{216}abc^4d^4;$$

$$(a+b+c)^3+1;\ \left(a+\frac{1}{a}\right)^6+\left(a-\frac{1}{a}\right)^6;\ \left(\frac{a^2}{bc}-\frac{b^2}{ac}\right)^3+\left(\frac{a3+b^3}{abc}\right)^3;$$

$\left(\frac{a+x}{a}\right)^3-\left(\frac{a}{a+x}\right)^3$; $\left(1-\frac{2}{x+1}\right)^6-\left(1+\frac{2}{x^3-1}\right)^6$;

$(x+y)^3+(2x+3y)^3$; $(a-1)^3+8$; $(1+x+x^2)^3-x^6$; $27(x+2)^3-64$;

$(x^2+y^2)^3-(3xy)^3$; $(x^2+4y^2)^6+(4x^2y^2)^3$;

$\left(a+x+\frac{x^2}{a}\right)^3-\left(x^2+\frac{x}{a}+\frac{1}{a^2}\right)^3$; $\left(\frac{x}{2}-\frac{y}{3}\right)^3+\left(\frac{y}{2}-\frac{x}{3}\right)^3$;

$\frac{1}{27}a^6b^6c^6-\frac{1}{8}(a^4bc+ab^4c+abc^4)^3$.

77. Any Binomial consisting of the difference of any the same powers of two quantities can always be resolved into two factors at least. Also a Binomial consisting of the sum of any the same *odd* powers of two quantities can always be resolved into two factors at least. See Div. page (29).

Ex. 1. $\frac{ab^6c}{32}-abc=abc\left(\frac{b^5}{32}-1\right)$

But $\frac{b^5}{32}-1=\left(\frac{b}{2}\right)^5-(1)^5$

$$=\left(\frac{b}{2}-1\right)\left\{\left(\frac{b}{2}\right)^4+\left(\frac{b}{2}\right)^3+\left(\frac{b}{2}\right)^2+\left(\frac{b}{2}\right)+1\right\}$$

$$=\frac{b}{2}-1\left\{\frac{b^4}{16}+\frac{b^3}{8}+\frac{b^2}{4}+\frac{b}{2}+1\right\}$$

Similarly $\frac{b^5}{32}+1$ may be resolved.

Ex. 2. $(x+1)^7+(y-1)^7=\{(x+1)+(y-1)\}\{(x+y)^6-(x+y)^5(y-1)+\&c.\}$

$$=(x+y)\{(\overline{x+y})^6-(x+y)^5(y-1)+\&c.\}$$

## Examples XXXVI.

Similarly the following may be resolved :—

$x^5\pm1$; $a^4b^4-c^8$; $x^{11}\pm1$; $a^7\pm128$; $x^5y^{10}\pm x^{10}y^5$; $(a+b)^4-a^2b^2$;

$(a+b)^5\pm(a-b)^5$; $\left(\frac{x}{2}+\frac{y}{3}\right)^5\pm\left(\frac{x}{3}+\frac{y}{2}\right)^{10}$; $\left(\frac{a^2}{b}+\frac{b^2}{a}\right)^6-c^6$;

$\left(\frac{bc}{a^2}\right)^5+\frac{a^5c^5}{b^{10}}$; $(2x+\frac{1}{2})^4-\left(\frac{x}{2}+2\right)^4$;

$(x^2+2xy+y^2)^7\pm(x^2-2xy+y^2)^7$; $\left(x^2+1+\frac{1}{x^2}\right)^3\pm\left(x^2-1+\frac{1}{x^2}\right)^3$;

$\left(\frac{x+1}{x-1}\right)^9+\left(\frac{x-1}{x+1}\right)^9$; $\left(1+x-\frac{1}{1-x}\right)^7\pm\left(1-x-\frac{1}{1+x}\right)^7$;

*Note.* The expression $x^5 \pm 1$ is read "$x$ to the fifth, plus or minus 1." $x^5+1$ is first to be resolved into factors, and then $x^5-1$.

**78.** By Multiplication we find that

$$(x^2+2xy+2y^2)\,(x^2-2xy+2y^2) = x^4+4y^4$$

Hence, any quantity of the form $x^4+4y^4$ can be resolved into two factors, which will be of the same form as the two factors given above.

Ex. 1.
$$\begin{aligned} x^4+4 &= x^4+4.1^4 \\ &= (x^2+2x+2)\,(x^2+2x-2). \end{aligned}$$

Ex. 2.
$$\begin{aligned} 16a^4+\frac{1}{4a^4} &= (2a)^4+4\left(\frac{1}{2a}\right)^4 \\ &= \left\{(2a)^2+2.(2a)\left(\frac{1}{2a}\right)+2\left(\frac{1}{2a}\right)^2\right\} \\ &\quad .\left\{(2a)^2-2(2a)\left(\frac{1}{2a}\right)+2\left(\frac{1}{2a}\right)^2\right\} \\ &= \left(4a^2+2+\frac{1}{2a^2}\right)\left(4a^2-2+\frac{1}{2a^2}\right) \end{aligned}$$

Ex. 3. $\left(\frac{x}{y}+\frac{y}{x}\right)^4+64\left(\frac{x}{y}-\frac{y}{x}\right)^4=\left(\frac{x}{y}+\frac{y}{x}\right)^4+4\left[2\left(\frac{x}{y}-\frac{y}{x}\right)\right]^4$

which reduces, by applying the same process as above, and simplifying each factor, to

$$\left(5\frac{x^2}{y^2}-2+\frac{y^2}{x^2}\right)\cdot\left(\frac{x^2}{y^2}-2+\frac{5y^2}{x^2}\right).$$

### Examples XXXVII.

Similarly the following may be resolved :—

$625a^8+4a^4b^4$ ; $x^4+64$ ; $81a^4+4b^4$ ; $\frac{x^4}{y^4}+4$ ; $(a+b)^4+4$ ;

$(a+b)^4+4(a-b)^4$ ; $(x^2+xy)^8+4\,(y^2+xy)^8$ ;

$(a+b+c)^4+4\,(a+b-c)^4$ ; $256\,(a^2+b^2)^4+324a^4b^4$ ;

$9\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^4+\frac{4}{9}\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^4$ ; $81\,(x^2+x+1)^4+4\,(x^2+x-1)^4$ ;

$625a^4\,(x^2+1)^4+4b^4\,(x^2-1)^4$ ; $10\,\frac{a^4b^4}{(x+y)^4}+40\,\frac{c^4}{(x-y)^4}$ ;

$\frac{1}{16}\left(\frac{x}{x-a}\right)^4+\frac{1}{4}\left(\frac{x+a}{x}\right)^4$ ; $6\left(\frac{a+b+c}{abc}\right)^4+\frac{1}{216}\left(\frac{a-b-c}{a^3}\right)^{12}$ ;

$\frac{324x^8}{(x^2-y^2)^4}+\frac{2500x^4}{(x-y)^4}$ ; $(3x^2+2x-1)^4+4\left(x^2-x+\frac{1}{2}\right)^4$ ;

$(x^2+3xy+2y^2)^4+\frac{1}{64}\,(x^2-3xy+2y^2)^4$.

## TRINOMIALS.

**79.** Any expression which is made up of the squares of two quantities together with twice the product of the same quantities is a perfect square, Art. 38, and consists therefore of two factors which are equal to each other.

Ex. 1.

$$\begin{aligned} 49x^2-14x+1 &= (7x)^2-2\times 7x\times 1+(1)^2 \\ &= (7x-1)^2 \end{aligned}$$

Ex 2.

$$\begin{aligned} \frac{a^2}{4}-\frac{ab}{3}+\frac{b^2}{9} &= \left(\frac{a}{2}\right)^2-2\cdot\frac{a}{2}\cdot\frac{b}{3}+\left(\frac{b}{3}\right)^2 \\ &= \left(\frac{a}{2}-\frac{b}{3}\right)^2 \end{aligned}$$

Ex. 3.

$$\begin{aligned} \frac{x^4}{4}-x^3+2x^2-2x+1 &= \left(\frac{x^4}{4}-x^3+x^2\right)+(x^2-2x)+1 \\ &= \left(\frac{x^2}{2}-x\right)^2+2\left(\frac{x^2}{2}-x\right)+(1)^2 \\ &= \left(\frac{x^2}{2}-x+1\right)^2 \end{aligned}$$

The pupil is recommended to turn back to the examples in the extraction of the square root, Art. 43, and to obtain the square roots by the method used in the above examples. When these have been done, he may try the following, as giving exercise in fractions.

### Examples XXXVIII.

$9a^2x^2-3abxy+\frac{b^2y^2}{4}$ ; $a^2b^2+ab+\frac{1}{4}$; $\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}$ ; $\frac{x^2}{9}+4xy+36y^2$ ;

$\frac{a^2}{4}-\frac{ab}{3}+\frac{b^2}{9}$ ; $\frac{x^4}{9}+\frac{2x^2}{21}+\frac{1}{49}$ ; $(a-b)^2+\frac{a-b}{3}+\frac{1}{36}$ ;

$49a^2x^2-2axy^2+\frac{1}{49}y^4$ ; $\frac{9}{4}x^4-2x^2y^2+\frac{4}{9}y^4$; $(a-1)^2+6(a-1)b+9b^2$;

$(x+1)^2+(10x+10)+25$ ; $(x+1)^2+3(x+1)\,(y+1)+\frac{9}{4}\,(y+1)^2$;

$\frac{16}{25}x^6-2x^3y^3+\frac{25}{16}y^6$ ; $9x^{10}-\frac{3}{2}\,x^7y^3+\frac{1}{16}x^4y^6$ ;

$5\frac{4}{9}\,a^2b^2+1\frac{13}{15}ab^2c+\frac{4}{25}b^2c^2$; $a^2+b^2+2ab+a+b+\frac{1}{4}$ ; $\frac{x^2}{81}-\frac{xy}{36}+\frac{y^2}{64}$ ;

$\frac{a^2+b^2}{4}+\frac{ab}{2}-\frac{a}{3}-\frac{b}{3}+\frac{1}{9}$ ; $\frac{a^4+1}{81}+\frac{2a^2}{81}-\frac{a^2c^2}{18}-\frac{a^2}{18}+\frac{c^4}{16}$.

**80.** In Art. (53) a method was given for resolving into two factors any trinomial of the form

$$x^2+(a+b)\,x+ab$$

A few more examples are here given, as applications of the same process :—

Ex. 1. $$x^4y^2z^2+9x^3y^2z^3-36x^2y^2z^4 = x^2y^2z^2\,(x^2+9xz-36z^2)$$
$$= x^2y^2z^2\,(x+12z)\,(x-3z).$$

Similarly the following may be resolved :—

### Examples XXXIX.

$a^6b^8+119a^7b^7-120a^8b^6$ ; $a^2b^4c^6-11ab^2c^3d^4-26d^8$ ;
$y^8+10ax^3y^4+21a^2x^6$ ; $a^6-8a^3bc^2+12b^2c^4$ ;
$a^6d^6-10a^3b^3c^3d^3-39b^6c^6$ ; $x^6y^4+x^3y^2z^2-20z^4$ ; $a^2b^2-8abc-65c^2$ ;
$7x^9+70x^6y^3+112x^3y^6$ ; $\frac{x^8}{3}-3x^4-\frac{10}{3}$ ; $\frac{x^4}{y^4}+\frac{x^2}{y^2}-30$ ; $\frac{a^2}{b^2}+13\frac{a}{b}+12$ ;
$\frac{a^2}{22}+\frac{4a}{11}-\frac{3}{2}$ ; $(x+1)^2+7\,(x^2-1)+6\,(x-1)^2$ ;
$\left(\frac{x+y}{x-y}\right)^4+1-56\left(\frac{x-y}{x+y}\right)^4.$

**81.** In the last case the co-efficient of the first term of the trinomial was $+1$ ; if, however, the co-efficient of the first term be different from 1, another method must be used, which we will now explain.

Let $ax^2+bx+c$ be the expression, $a\,b\,c$ having no common factor. Multiply it by $a$, it then becomes

$$a^2x^2+abx+ac.$$

Let us assume that this is equal to the product of the two factors,

$$(ax+m),\text{ and }(ax+n).$$

Multiplying these together, they give

$$a^2x^2+a\,(m+n)\,x+mn.$$

Now this must be the same as the original quantity,

$$a^2x^2+abx+ac,$$

$$\begin{array}{lrl} \therefore & a\,(m+n) = ab & \\ \text{Or} & m+n = b & \left.\begin{array}{l}(1)\\(2)\end{array}\right. \\ \text{And} & mn = ac & \end{array}$$

We must now find by trial what values $m$ and $n$ must have in (1) and (2), and then substitute these values of $m$ and $n$ in the two factors $(ax+m)$ and $(ax+n)$.

Let each of these now be divided by the greatest number which is contained in it without remainder, the resulting quantities will be the factors required.

Ex. 1. Let $14x^2+25x+6$ be the expression.

Multiply this by 14, it becomes

$$196x^2+350x+84.$$

Let $(14x+m)$ and $(14x+n)$ be the two factors,

$$\therefore \quad 14(m+n) = 350$$

$$\left.\begin{array}{ll}\text{Or} & m+n = 25 \\ \text{And} & mn = 84\end{array}\right\}$$

Hence we find by trial that

$$\left.\begin{array}{l} m = 21 \\ n = 4 \end{array}\right\}$$

Substituting for $m$ and $n$ their values,

$14x+m$ becomes $14x+21$

and $14x+n$ ,, $14x+4$.

The first of these divides by 7, and the next by 2,

Hence $(2x+3)$ and $(7x+2)$ are the two factors;

Ex. 2. Let $30x^2-83x+8$ be the trinomial.

Multiply by 30, it becomes

$$900x^2-2490x+240,$$

Also $(30x+m)$ and $(30x+n)$ are the two factors;

$$\therefore \quad 30(m+n) = -2490$$

$$\left.\begin{array}{ll}\text{Or} & m+n = -83 \\ & mn = 240\end{array}\right\}$$

Hence we find $\left.\begin{array}{l} m = -80 \\ n = -3 \end{array}\right\}$

The factors are

$$\left.\begin{array}{ll} & 30x-80 \\ \text{and} & 30x-3 \end{array}\right\}$$

10 divides the first, and 3 divides the second,

$\therefore$ $(3x-8)$ and $(10x-1)$

are the factors required.

Ex. 3. Let $12x^2-\frac{x}{2}-\frac{1}{1}$ be the trinomial;

Then $$12x^2-\frac{x}{2}-\frac{1}{4}=\frac{48x^2-2x-1}{4}.$$

And we must find the two factors of $48x^2-2x-1$ by the method given in the two former examples;

These will be found to be $(8x+1)$ and $(6x-1)$,

$$\therefore \quad 12x^2-\frac{x}{2}-\frac{1}{4}=\frac{(8x+1)(6x-1)}{4}$$

## Examples XL.

$2x^2+11x+12$; $21x^2-50x+21$; $33x^2+133x+4$; $30x^2+29x-45$; $56x^2+3x-9$; $22x^4-57x^2+36$; $210x^4+x^2-171$; $30a^2+ab-14b^2$; $36x^2-141xy-65y^2$; $110x^4-x^2y^2-y^4$; $16x^4+45x^2y-9y^2$; $49a^6+394a^3b+16b^2$; $6a^8-5a^4b^4-99b^8$; $72a^4b^4-5a^2b^2c^2d^2-12c^4d^4$; $24a^4-64a^3b+32a^2b^2$; $21a^2+105ab+126b^2$; $18a^2x^2-9abxy-14b^2y^2$; $9x^{12}-24x^6y^4-9y^8$; $100x^4y^2-9x^2y^2z^2-91y^2z^4$; $4m^2+2mn-110n^2$; $84y^4+87x^2y^2-162x^4$; $36m^4x^4-237m^2n^2x^2y^2+126x^4y^4$;

$$\frac{x^2}{2}-\frac{3}{4}(5x+9);\ x^4y^2+x^2y^4-\frac{113}{56}x^3y^3;\ x^6+y^6+\frac{2321}{49};$$

$$\frac{9}{4}a^2b^2+\frac{13}{6}abcd+\frac{4}{9}c^2d^2;\ a^4b^4-\frac{65}{108}a^2b^6-\frac{1}{9}b^8;\ \frac{11}{9}x^6+\frac{5}{36}x^3y^3-\frac{1}{44}y^6;$$

$$2x^2-\frac{19}{3}x+1;\ 21x^2-\frac{x}{4}-\frac{1}{8};\ a^4+\frac{577}{24}a^2b^2+\frac{b^4}{24};\ \frac{1}{6}a^2+\frac{29}{54}ab+\frac{1}{9}b^2;$$

$$a^4+\frac{577}{24}a^2b^2+\frac{6^4}{24};\ \frac{1}{6}a^2 \qquad ;\ \frac{1}{11}x^2-\frac{2}{33}xy^2+y^2;$$

$$\frac{p^4}{49}+\frac{p^2}{42}+\frac{1}{6};\ \frac{25}{36}p^6+\frac{5}{6}p^3q-2q^2;\ a^2b^2+\frac{28}{9}ab+\frac{1}{3};$$

$(a-b)^2-7(a-b)+8$.

*Note.*—This expression is of the form

$$a^2-7a+8$$

which equals $(a-7)(a-1)$;

Hence the factors of the given expression are

$$\{(a-b)-1\}\{(a-b)-7\}$$
$$=(a-b-1)(a-b-7)$$

$12(a+b)^2-11(a+b)(c+d)+2(c+d)^2$;

$21(a+b)^2+20(a+b)c+4c^2$; $6(x^2+x)^2-(x^2+x)(x+1)+(x+1)^2$;

$$(x-2)^2-\frac{13}{2}(x-2)(y-2)+(y-2)^2;$$

$$\frac{1}{3}(x^2+1)^2+\frac{55}{72}(x^2+1)(x+1)-\frac{1}{3}(x+1)^2;$$

$$\frac{3}{4}\left(\frac{x}{2}+\frac{y}{3}\right)^2-\frac{25}{12}\left(\frac{x}{2}+\frac{y}{3}\right)\left(\frac{x}{3}+\frac{y}{2}\right)+\frac{4}{3}\left(\frac{y}{2}+\frac{x}{3}\right).$$

It will be shown hereafter that the expression $ax^2+bx+c$ cannot be roken into factors if $b^2$ be less than $4ac$. When therefore an expreson of that form is proposed to be resolved into its factors, it is derable to apply this test, as it will be useless to proceed further if $b^2$ not greater than or equal to $4ac$. For instance, the expressions $+4x+5$, and $10x^2-6x+1$, are not resolvable into factors. If the udent will work out these two examples by the rules before given, e will find it impossible to get values for *m* and *n*.

**82.** Expressions which contain higher powers than the second of $x$ may however be sometimes broken into factors, although the condition just mentioned be not satisfied, as will be seen in the following Article.

Any trinomial of the form $x^4+a^2x^2+a^4$ may be resolved into two factors as follows :—

$$\begin{aligned} x^4+a^2x^2+a^4 &= x^4+2a^2x^2+a^4-a^2x^2 \\ &= (x^2+a^2)^2-(ax)^2 \\ &= (x^2+a^2+ax)\ (x^2+a^2-ax) \end{aligned}$$

Or $\quad x^4+a^2x^2+a^4 = (x^2+ax+a^2)\ (x^2-ax+a^2)$

Similarly may $x^4-7a^2x^2+a^4$ be resolved

Also $\quad x^4-14a^2x^2+a^4$

And $\quad a^4x^4-23a^2x^2+1$,

for each of these may be expressed as the difference of two squares, thus :

$$\begin{aligned} x^4-7a^2x^2+a^4 &= (x^4+2a^2x^2+a^4)-9a^2x^2 \\ &= (x^2+a^2)^2-(3ax)^2 \\ &= (x^2+3ax+a^2)\ (x^2-3ax+a^2). \end{aligned}$$

## Examples XLI.

$4x^4+4a^2x^2+4a^4$; $81x^4+9a^2x^2+a^4$; $x^4+9x^2+81$;
$16x^4+36x^2y^2+81y^4$; $16m^8+4m^4n^4+n^8$;
$81\ (m+1)^4+225\ (m+1)^2+625$; $(x-y)^4+(x^2-y^2)^2+(x+y)^4$;
$\dfrac{x^4}{16}+\dfrac{x^2y^2}{64}+\dfrac{y^4}{256}$; $1+\dfrac{x^4}{9}+\dfrac{x^8}{81}$; $16a^8+a^6+\dfrac{a^4}{16}$;
$625x^4+400x^2y^2+256y^4$; $\dfrac{16x^8}{625}+\dfrac{36x^4}{625}+\dfrac{81}{625}$; $(x+1)^4+(x+1)^2+1$;
$x^4+5a^2x^2+9a^4$; $4x^8-16a^4x^4+9a^8$; $81+440x^2+2401x^4$;
$256x^8-4x^6+x^4$; $\dfrac{16}{81}y^4-194x^2y^2+\dfrac{81}{16}x^4$;
$(x+1)^4-\dfrac{23}{72}(x+1)^2+\dfrac{1}{256}$; $\dfrac{2401}{10000}x^{12}+\dfrac{257x^6}{100}+81y^{12}$;
$6561a^4+\dfrac{b^2}{625}(23a^2+16b^2)$.

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## MULTINOMIALS.

**83.** It has been shown that—

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

And, $\quad (a-b)^3 = a^3-3a^2b+3ab^2-b^3.$

If, therefore, a quantity of 4 terms can be made to appear in either of these forms, it must be a perfect cube, or it consists of three equal binomial factors.

Again, $27a^3+27a^2+9a+1$ takes the form

$$(3a)^3+3\,.\,(3a)^2\,.\,1+3\,(3a)\,.\,1^2+1^3,$$

And it therefore equals $(3a+1)^3$.

Ex. 2. $\dfrac{(a+2b)^3}{8}+\frac{9}{4}(a+2b)^2\,(c+d)+\frac{27}{2}\,(a+2b)\,(c+d)+27\,(c+d)^3$

$$=\left(\frac{a+2b}{2}\right)^3+3\left(\frac{a+2b}{2}\right)^2\,\overline{(3c+d)}+3\left(\frac{a+b}{2}\right)(3\,\overline{c+d})^2+(3\,\overline{c+d})^3$$

$$=\left(\frac{a+2b}{2}+3\overline{c+d}\right)^3=\left(\frac{a+2b+6c+6d}{2}\right)^3.$$

Similarly the following may be resolved :—

## Examples XLII.

$a^3+3a^2b^2+3ab^4+b^6$ ; $8x^3-36x^2y+54xy^2-27y^3$ ;
$4a^5+12a^4+12a^3+4a^2$ ; $27\,(c^3x^3-bc^2x^2y+\frac{1}{3}b^2cxy^2)-b^3y^3$ ;
$\dfrac{a^3}{8}-\dfrac{b^3}{27}-\dfrac{ab}{2}\left(\dfrac{a}{2}-\dfrac{b}{3}\right)$ ; $\dfrac{x^6+9x^4y^2+27x^2y^4+27y^6}{64}$ ;
$1000\,a^9b^6c^3-30a^6b^4c^2+\frac{3}{10}\,a^3b^2c-\frac{1}{1000}$ ;
$(a+b)^3+6\,(a+b)^2\,c+12\,(a+b)\,c^2+8c^3$ ; $\frac{1331}{1728}\,a^9-\frac{11}{4}\,a^6b^3+\frac{33}{11}\,a^3b^6$
$-\frac{1728}{1331}\,b^9$ ; $125\,(x^3+y^3)^3+375xy\,(x+y)$ ; $\dfrac{x}{81}\,(3x^2+y^2)$
$-\dfrac{y^2}{729}\,(27x+y)$ ; $\frac{1}{8}\,(a^2+b^2)^2\,(a-b)^2+\dfrac{a^2b^3}{54}\,(9a^2+9b^2-2ab)$.

**84.** When a quantity of four terms is equal to the sum or difference of two cubes, it may be broken into factors, as in the following examples :—

Ex. 1.
$$\begin{aligned} a^3-3a^2+3a-7 &= (a^3-3a^2+3a-8)+1\\ &= (a-2)^3+1^3\\ &= (\overline{a-2}+1)\,(\overline{a-2}|^2-\overline{a-2}\,.\,1+1^2)\\ &= (a-2)\,(a^2-5a+7). \end{aligned}$$

Ex. 2.
$$\begin{aligned} x^6+12x^4y+48x^2y^2-61y^3 &= (x^2+4y)^3-(5y)^3\\ &= (x^2-y)\,(x^4+13x^2y+61y^2), \end{aligned}$$

After the necessary reductions.

## Examples XLIII.

$64x^3-48x^2+12x+26$ ; $2x^3+3x^2+3x+1$ ;
$9x^4\,(x^2+1)+27\,(x^2+3)$ ; $x^9-4x^6y^3+3x^3y^6-y^9$ ;

$-117x^3+36x^2y+54xy^2+27y^3$; $\frac{9}{8}x^3-\frac{3x^2}{4}+\frac{3x}{2}-1$;
$512a^9+b^9-24a^3b^3(a^3-b^3)$; $x^9+\frac{64}{3}x^3-\frac{512}{27}$;
$125x^3-\frac{75}{2}x^2+\frac{15}{4}x+\frac{7}{8}$; $a^6+3a^4-a^3-9a^2-48a-63$;
$\frac{1}{216}(8a^3-35b^3)-\frac{ab}{36}(2a-b)$; $296x^6+1344x^2y^4+1176x^4y^2$
$+343x^6$; $\frac{a^6}{343}+\frac{3a^4b^2}{196}+\frac{3a^2b^4}{112}-\frac{63}{64}b^6$; $\frac{(a+b)^3}{27}-9(a-b)^3$
$-(a^2-b^2)\left(\frac{a+b}{3}+a-b\right)$.

**85.** It may be found by actual multiplication, that—

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

Hence we derive the method of resolving into two factors a quantity made up of the sum of the cubes of three quantities, and three times the product of the same three quantities taken negatively.

Ex. 1. $a^3+b^3-c^3+3abc=a^3+b^3+(-c)^3-3ab(-c)$
$=(a+b-c)(a^2+b^2+c^2-ab+ac+bc)$.

Ex. 2. $a^3-b^3-c^3-3abc=a^3+(-b^3)+(-c)^3-3a(-b)(-c)$
$=(a-b-c)(a^2+b^2+c^2-ab-ac+bc)$.

Ex. 3. $a^3+b^3-3ab+1$; $x^3+y^3-6xy+8$; $x^3+y^3+3xy-1$;
$8a^3-b^3-c^3-6abc$; $64(x^3y^3+x^3z^3-y^3z^3+3x^2y^2z^2)$.

Ex. 4. $27a^6-10a^3+1=(3a^2)^3+(2a)^3+1^3-3(3a^2)(2a).1$
$=(3a^2+2a+1)(\overline{3a^2}\big|^2+\overline{2a}\big|^2+1^2-3a^2.2a$
$-3a^2.1-2a.1)$
$=(3a^2+2a+1)(9a^4-6a^3+a^2-2a+1)$.

## Examples XLIV.

$x^6y^3+y^6z^3-x^6z^3+3x^4y^3z^2$; $8x^3y^3-27y^3z^3+64x^3z^3+72x^2y^2z^2$;
$8abcxyz-a^3x^3-64c^3z^3-8b^3y^3$; $8a^3-b^3-3ab-\frac{1}{8}$;
$\frac{8}{27}a^3-\frac{1}{8}b^3-\frac{27}{8}c^3+\frac{3}{2}abc$; $8x^5(8x^4+8xy^3+3y)-\frac{1}{8}$;
$\frac{x^6}{a^6}+\frac{y^6}{b^6}+\frac{z^6}{c^6}-3\frac{x^2y^2z^2}{a^2b^2c^2}$; $8a^6+4a^3b^3+b^6$; $\frac{a^6}{b^6}+4\frac{a^3}{b^3}-1$;
$\frac{a^9}{b^9}+2\frac{a^6}{b^6}-\frac{a^3}{b^3}$; $a^3+\frac{1}{a^3}+18$; $(a-b)^3-(a-c)^3-(a-d)^3$
$-3(a-b)(a-c)(a-d)$; $\frac{x^9}{y^9}+\frac{7}{8}\frac{x^6}{y^6}-\frac{x^3}{8y^3}$;
$\frac{a^9}{b^6c^3}-\frac{b^9}{a^6c^3}-\frac{c^9}{a^6b^3}+3$.

**86.** By actual multiplication, it may be seen that—

$$a^2b^2+a^2d^2+b^2c^2+c^2d^2 = (a^2+c^2)\,(b^2+d^2) \qquad (1)$$

Also, $$a^2b^2+a^2d^2-b^2c^2-c^2d^2 = (a^2-c^2)\,(b^2+d^2)$$
$$= (a-c)\,(a+c)\,(b^2+d^2) \qquad (2)$$

And $$a^2b^2-a^2d^2-b^2c^2+c^2d^2 = (a^2-c^2)\,(b^2-d^2)$$
$$= (a-c)\,(a+c)\,(b-d)\,(b+d) \qquad (3)$$

Hence it appears that when each of the four terms of an expression is the square of the product of two of four letters, each letter occurring twice only, and the signs are either all the same, or two of them positive and two negative, it may be broken into *two*, *three*, or *four* factors, according as it is of the form (1), (2), or (3).

The following examples fall under one of the above forms;—

### Examples XLV.

$a^2c^2+9a^2d^2+4b^2c^2+36b^2d^2$; $x^2y^2+16b^2x^2+a^2y^2+16a^2b^2$; $a^2x^2-4b^2x^2-4a^2y^2+16b^2y^2$; $a^2x^2+b^2x^2+a^2y^2+b^2y^2$; $c^2x^2-d^2x^2-a^2b^2c^2+a^2b^2d^2$; $36a^2x^2+64a^2y^2-81b^2x^2-144b^2y^2$; $100\,(p^2x^2+q^2y^2)-p^2y^2-10000q^2x^2$; $36a^2b^2c^2d^2+16x^2y^2z^2v^2-64a^2b^2z^2v^2-9c^2d^2x^2y^2$; $9p^4m^2-144p^4n^2+36q^4m^2-576q^4n^2$;

$$\frac{a^2b^2}{36}+\frac{c^2d^2}{400}-\frac{a^2c^2}{64}-\frac{b^2d^2}{225};\ \frac{9x^2z^2}{4}-1-\frac{4y^2}{9x^2z^2}+81.$$

**87.** Since—

$$(a+b\pm c)^2 = a^2+b^2+c^2+2ab\pm 2ab\pm 2bc$$

it appears that a quantity consisting of the squares of three terms and twice the product of every two of them, is a perfect square, and can therefore be resolved into two equal factors.

*Note.* The squares must all have the positive sign, and the products must either be all positive, or two of them must be negative and the other positive.

### Examples XLVI.

$x^2+y^2+4z^2+2xy+4xz+4yz$; $9x^4-6x^2y^2-6x^2+y^4+2y^2+1$; $49a^2b^2+9a^2c^2+81b^2c^2-42a^2bc-126ab^2c+54abc^2$;

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}-\frac{xy}{3}-\frac{xz}{1}+\frac{yz}{6};$$

$$x^4+x^3+\frac{3}{4}x^2+\frac{x}{4}+\frac{1}{16}.$$

This last example may be put in the form—

$$x^4+\frac{x^2}{4}+\frac{1}{16}+x^3+\frac{x^2}{2}+\frac{x}{4}$$

$$\frac{x^4}{4y^4}+3\frac{x^2}{y^2}+4-\frac{x^3}{y^3}-\frac{4x}{y}$$

$$\frac{a^2}{b^2}+\frac{c^3}{d^2}+\frac{e^2}{f^2}+\frac{2ac}{bd}-\frac{2ae}{bf}-\frac{2ce}{df}$$

$$\left(\frac{a+b}{a-b}\right)^2+3+\left(\frac{a-b}{a+b}\right)^2+2.\frac{a+b}{a-b}+2.\frac{a-b}{a+b}.$$

**88.** An expression of the form—

$$a^2+b^2+mc^2+2ab\pm 2nac\pm 2nbc,$$

can sometimes be resolved into factors; for it equals

$$(a+b\pm nc)^2-(n^2-m)\,c^2,$$

and if $(n^2-m)$ be a square number or quantity, the expression is the difference of two squares.

Suppose $\qquad n^2-m=p^2$

it then becomes $\qquad (a+b\pm nc)^2-p^2c^2$

$$=(a+b\pm nc+pc)\,(a+b\pm nc-pc).$$

Ex. 1. $\qquad 9a^2b^2+16a^2c^2-15b^2c^2+24a^2bc-42ab^2c-56abc^2$

$$=(3ab+4ac-7bc)^2-(8bc)^2$$

$$=(3ab+4ac+bc)\,(3ab+4ac-15bc).$$

### Examples XLVII.

$a^2+b^2-5c^2-2ab+4ac-4bc$;

$4y^2+z^2-7x^2-12xy-6xz+4yz$;

$4x^2\,(y^2+2z^2-3yz)+5yz\,(5yz+4xy-6xz)$;

$\frac{x^2}{y^2}-21\frac{y^2}{z^2}-4\frac{x}{z}\left(\frac{x}{z}-\frac{x}{y}+1-2\frac{y}{z}\right)$;

$9a\,(9a^2-15b^2+c^2-42ab+6ac+14bc)$;

$\frac{1}{9}\,(2x^2+y^2+z^2+3xy-3xz-2yz)$;

$\frac{x^4}{b^2c^2}-48\frac{y^4}{a^2c^2}-8\frac{x^2y^2}{abc^2}+32\frac{z^2}{ab}\left(\frac{8z^2}{ab}+\frac{x^2}{bc}-\frac{4y^2}{ac}\right)$;

$a^2\left(\frac{1}{(a+x)^2}+\frac{2}{a^2-x^2}\right)+a^4\left(\frac{1}{(a^2-x^2)^2}-\frac{2}{(a+x)\,(a^2-x^2)}-\frac{2}{(a-x)\,(a^2-x^2)}\right)$; $\frac{1}{a^2x^2}+\frac{4}{b^2y^2}+\frac{5}{c^2z^2}-\frac{4}{abxy}-\frac{6}{acxz}+\frac{12}{bcyz}$; $x^2+y^2+2xy+2xz+2yz$;

$\frac{4x^2}{a^2}+\frac{z^2}{4c^2}-\frac{4xy}{ab}-\frac{2xz}{ac}+\frac{yz}{bc}.$

**89.** The expression—

$$\begin{aligned} &a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2 \quad (1) \\ &= (b^2+c^2-a^2)^2 - 4b^2c^2 \\ &= \{(b+c)^2-a^2\}\{(b-c)^2-a^2\} \\ &= (b+c+a)(b+c-a)(b-c-a)(b-c+a). \end{aligned}$$

Similarly it may be shown that—

$$\begin{aligned} &a^4+b^4+c^4+2a^2b^2+2a^2c^2-2b^2c^2 \\ &= (a^2+b^2+2bc+c^2)(a^2+b^2-2bc+c^2). \end{aligned}$$

Also, $a^4+b^4+c^4-2a^2b^2+2a^2c^2+2b^2c^2 = (b^2+c^2-a^2)^2$;

And $a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2 = (a^2+b^2+c^2)^2$.

Hence it appears that a quantity consisting of six terms of the same form as in (1) may always be resolved into factors, whatever be the signs of the last three terms.

Resolve the following into two or four factors.

### Examples XLVIII.

$$\begin{gathered} a^4+4b^4+c^4-4a^2b^2-2a^2c^2-4b^2c^2; \\ a^4+4b^4+c^4+4a^2b^2-2a^2c^2-4b^2c^2; \\ a^4+4b^4+c^4+4a^2b^2+2a^2c^2-4b^2c^2; \\ a^4+4b^4+c^4+4a^2b^2+2a^2c^2+4b^2c^2. \end{gathered}$$

Let all the following examples be taken with the same variations of sign as have been given in the above example.

$$2x^2y^2+2x^2z^2+2y^2z^2-x^4-y^4-z^4;\ 2x^2y^2+2x^2+2y^2-x^4-y^4-1;$$

$$-12p^2q^2+16p^2r^2+24q^2r^2-4p^4-9q^4-16r^4;$$

$$\frac{1}{x^4}+\frac{4}{y^4}+\frac{9}{z^4}-\frac{4}{x^2y^2}+\frac{6}{x^2z^2}+\frac{12}{y^2z^2};$$

$$24a^2b^2x^2y^2-30a^2c^2x^2z^2-40b^2c^2y^2z^2-9a^4x^4-16b^4y^4-25c^4z^4;$$

$$\frac{p^2q^2}{8a^2b^2c^4}+\frac{p^2r^2}{2a^2b^4c^2}+\frac{q^2r^2}{2a^4b^2c^2}-\frac{p^4}{16b\ c^4}-\frac{q^4}{16a^4c^4}-\frac{r^4}{a^4b^4}.$$

## CHAPTER XI.

### FRACTIONAL AND NEGATIVE INDICES, AND SURDS.

**90.** In such expressions as $a^3$, $(a+b)^2$, $x^my^n$, &c., the small figure, or index, at the right hand of each quantity denotes the number of factors in it; that is—

$$a^3 = a \, . \, a \, . \, a$$

$$x^m y^n = x \, . \, x \, . \, x \, . \, . \, . \, . \, . \, . \text{ continued to } m \text{ factors.}$$

$$\times \; y \, . \, y \, . \, y \, . \, . \, . \, . \, . \, . \text{ continued to } n \text{ factors.}$$

When, therefore, the index is a positive whole number, its meaning is definite, but the definition in Art. (11) does not decide the meaning of a fractional or negative index as $x^{\frac{1}{2}}$, $x^{\frac{2}{3}}$, $x^{-3}$, &c. These cases we shall now consider.

Before we proceed to assign a meaning to fractional and negative indices, it will be necessary to assume that these obey the same law in multiplication and division as positive integral indices have been already proved to do: Art. (31) and Art. (35), or,

1. Different powers of the same letter or quantity are multiplied together by *adding* the indices of all the factors, or, $x^m . x^n = x^{m+n}$ is always true, whatever $m$ and $n$ may be.

2. Different powers of the same letter or quantity are divided by *subtracting* the index of the divisor from that of the dividend,

or, $x^m \div x^n = x^{m-n}$ is always true, whatever $m$ and $n$ may be, integral or fractional, positive or negative.

**91.** To interpret the expression $x^{\frac{1}{2}}$

Now, $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$ by (1)

Again, $\sqrt{x} \times \sqrt{x} = x$ by def. of square root.

$$\therefore \quad (x^{\frac{1}{2}})^2 = (\sqrt{x})^2.$$

Taking the square root of both sides, we have—

$$x^{\frac{1}{2}} = \sqrt{x}.$$

Similarly we may proceed with $x^{\frac{1}{3}}$

For $\quad x^{\frac{1}{3}} . x^{\frac{1}{3}} . x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$

$$= x$$

And $\quad \sqrt[3]{x} . \sqrt[3]{x} . \sqrt[3]{x} = x$ by definition

$$\therefore \quad (x^{\frac{1}{3}})^3 = (\sqrt[3]{x})^3.$$

Taking the cube root of both sides, we have—

$$x^{\frac{1}{3}} = \sqrt[3]{x}.$$

Thirdly, to find the meaning of $x^{\frac{1}{m}}$, $m$ being any positive integer.

Now, $x^{\frac{1}{m}} . x^{\frac{1}{m}} . x^{\frac{1}{m}} \ldots$ to $m$ factors.

$$= x^{\frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \ldots \text{ to } m \text{ terms.}}$$

$$= x \text{ by (1)}.$$

Again, $\sqrt[m]{x} . \sqrt[m]{x} . \sqrt[m]{x} \ldots$ to $m$ factors $= x$ by def.

$$\therefore \quad \left(x^{\frac{1}{m}}\right)^m = \left(\sqrt[m]{x}\right)^m.$$

Taking the $m^{th}$ root of both sides, we have—

$$x^{\frac{1}{m}} = \sqrt[m]{x}.$$

Hence it appears that we may define a fractional index as follows:—

*The index* $\frac{1}{2}$ *denotes the square root*
$\frac{1}{3}$ ,, *cube root*

and generally

*The index* $\frac{1}{m}$ *denotes the* $m^{th}$ *root.*

**92.** To interpret the expression $x^{\frac{2}{3}}$.

Now, $x^{\frac{2}{3}} . x^{\frac{2}{3}} . x^{\frac{2}{3}} = x^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3}}$

$$= x^2.$$

But $\sqrt[3]{x^2} . \sqrt[3]{x^2} . \sqrt[3]{x^2} = x^2$ by def. of the cube root.

$$\therefore \quad \left(x^{\frac{2}{3}}\right)^3 = \left(\sqrt[3]{x^2}\right)^3$$

$\therefore \quad x^{\frac{2}{3}} = \sqrt[3]{x^2}$ by taking the cube root of both sides.

Similarly it may be shown that—

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

where $^p$ and $^q$ are any positive integers.

*Hence it appears that* $x^{\frac{2}{3}}$ *means the cube root of the square of* $x$ *:*

*and generally* $x^{\frac{p}{q}}$ *denote the* $q^{\text{th}}$ *root of the* $p^{\text{th}}$ *power of* $x$.

*Therefore, to express the* $n^{\text{th}}$ *root of any power of a quantity, divide the index of the power by* $n$.

**93.** Next, to consider the meaning of a negative index, as $x^{-3}$, $x^{-q}$, &c.

First, take the expression $x^{-3}$.

$$\text{Now,}\qquad x^5 \times x^{-3} = x^{5-3} \text{ by (1)}$$
$$= x^2.$$

$$\text{Again,}\qquad x^5 \times \frac{1}{x^3} = x^5 \div x^3$$
$$= x^{5-3}$$
$$= x^2 \text{ by (2)}$$

$$\therefore\qquad x^{-3} \text{ is the same as } \frac{1}{x^3}$$

$$\text{Again,}\qquad x^p \times x^{-q} = x^{p-q}$$

$$\text{And}\qquad x^p \times \frac{1}{x^q} = x^p \div x^q$$
$$= x^{p-q}$$

$$\text{Or,}\qquad x^{-q} \text{ is the same as } \frac{1}{x^q}.$$

*Hence it is always possible in fractional expressions to transfer any power of a quantity from the numerator to the denominator; and* vice versâ, *by changing the sign of its index.*

$$\text{Thus}\qquad \frac{x^5y^{-2}}{x^3y^4} = x^5y^{-2}.x^{-3}y^{-4} = x^{5-3}.y^{-2-4}$$
$$= x^2y^{-6}.$$

**94.** The following propositions in Indices should be carefully studied.

*Note.*—In the next eight propositions, the indices are all positive.

i. To prove $(x^m)^n = x^{mn} = (x^n)^m$,

$m$ and $n$ being integers.

For $(x^m)^n = x^m.x^m.x^m \dots$ &c. to $n$ factors.

$$= x^{m+m+m+\dots\dots\text{ to } n \text{ terms.}}$$

$$= x^{mn}.$$

Similarly, it may be shown that—

$$(x^n)^m = x^{mn}$$

$$\therefore \quad (x^m)^n = (x^n)^m.$$

Ex. $(x^3)^2 = (x^2)^3 = x^6.$

ii. To prove $\left(x^{\frac{1}{n}}\right)^m = (x^m)^{\frac{1}{n}} = x^{\frac{m}{n}}.$

For $\left(x^{\frac{1}{n}}\right)^m = x^{\frac{1}{n}}.x^{\frac{1}{n}}.$ &c. to $m$ factors

$$= x^{\frac{1}{n}+\frac{1}{n}+\text{ \&c. to } m \text{ terms.}}$$

$$= x^{\frac{m}{n}}.$$

And $\left\{(x^m)^{\frac{1}{n}}\right\}^n = (x^m)^{\frac{1}{n}}.(x^m)^{\frac{1}{n}} \dots$ &c. to $n$ factors

$$= (x^m)^{\frac{1}{n}+\frac{1}{n}+\text{ \&c.} \dots \text{ to } n \text{ terms.}}$$

$$= (x^m)^{\frac{n}{n}}.$$

Or, $\left\{(x^m)^{\frac{1}{n}}\right\}^n = x^m.$

Taking the $n^{\text{th}}$ root of both sides, we get—

$$(x^m)^{\frac{1}{n}} = x^{\frac{m}{n}}$$

$$\therefore \quad \left(x^{\frac{1}{n}}\right)^m = x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$$

Ex. $\left(x^{\frac{1}{2}}\right)^4 = x^{\frac{4}{2}} = (x^4)^{\frac{1}{2}} = x^2.$

iii. To prove $\left(x^{\frac{1}{m}}\right)^{\frac{1}{n}} = x^{\frac{1}{mn}} = \left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}.$

Begin by raising $\left(x^{\frac{1}{n}}\right)^{\frac{1}{n}}$ to the $\overline{mn}|^{\text{th}}$ power and proceed as in (ii.)

Ex. $\left(x^{\frac{1}{3}}\right)^{\frac{1}{4}} = x^{\frac{1}{12}} = \left(x^{\frac{1}{4}}\right)^{\frac{1}{3}} = \sqrt[12]{x}.$

iv. To prove $x^{\frac{1}{m}}.y^{\frac{1}{m}} = (xy)^{\frac{1}{m}}.$

For $\left(x^{\frac{1}{m}}.y^{\frac{1}{m}}\right)^m = x^{\frac{1}{m}}.y^{\frac{1}{m}} \times x^{\frac{1}{m}}.y^{\frac{1}{m}} \times$ &c. ... to $2m$ factors,

$$= \left(x^{\frac{1}{m}}.x^{\frac{1}{m}} \ldots . \text{ to } m \text{ factors}\right)$$

$$\times \left(y^{\frac{1}{m}}.y^{\frac{1}{m}} \ldots \text{ to } m \text{ factors}\right)$$

$$= xy.$$

And $\left\{(xy)^{\frac{1}{m}}\right\}^{m} = (xy)^{\frac{1}{m}}.(xy)^{\frac{1}{m}} \ldots \text{ to } m \text{ factors.}$

$$= (xy)^{\frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \&c. \ldots \text{ to } m \text{ terms.}}$$

$$= xy$$

$$\therefore \quad \left(x^{\frac{1}{m}}.y^{\frac{1}{m}}\right)^{m} = \left\{(xy)^{\frac{1}{m}}\right\}^{m}$$

Taking the $m^{\text{th}}$ root of both sides, we have—

$$x^{\frac{1}{m}}y^{\frac{1}{m}} = (xy)^{\frac{1}{m}}$$

Ex. $x^{\frac{1}{3}}.y^{\frac{1}{3}} = (xy)^{\frac{1}{3}}$

Or, $\sqrt[3]{x}.\sqrt[3]{y} = \sqrt[3]{xy}.$

Similarly it may be shown that—

$$x^{\frac{1}{m}}y^{\frac{1}{m}}z^{\frac{1}{m}} = (xyz)^{\frac{1}{m}},$$

and the proof may be extended to any number of factors.

v. To prove $x^{\frac{1}{m}} \div y^{\frac{1}{m}} = \left(\frac{x}{y}\right)^{\frac{1}{m}}.$

Raise $\dfrac{x^{\frac{1}{m}}}{y^{\frac{1}{m}}}$ to the $m^{\text{th}}$ power and proceed as in (iv.)

Ex. $\dfrac{x^{\frac{1}{4}}}{y^{\frac{1}{4}}} = \left(\dfrac{x}{y}\right)^{\frac{1}{4}}$

Or, $\dfrac{\sqrt[4]{x}}{\sqrt[4]{y}} = \sqrt[4]{\dfrac{x}{y}}.$

vi. To prove $x^{\frac{m}{n}} = x^{\frac{mp}{np}}.$

Raise $x^{\frac{m}{n}}$ to the $\overline{np}^{\text{th}}$ power, and then raise $x^{\frac{mp}{np}}$ to the $\overline{np}^{\text{th}}$ power; these results will be seen to be equal: take the $\overline{np}^{\text{th}}$ root of both sides, and the results will be equal.

Similarly it might be shown that—

$$x^{\frac{\frac{m}{p}}{\frac{n}{p}}} = x^{\frac{m}{n}}.$$

Ex. $\quad x^{\frac{2}{3}} = x^{\frac{4}{6}} = x^{\frac{2n}{3n}}.$

Or, $\quad \sqrt[3]{x^2} = \sqrt[6]{x^4} = \sqrt[3n]{x^{2n}}.$

*Note.*—It should be carefully noticed that this proposition is not self-evident; for $x^{\frac{m}{n}}$ means the $n^{\text{th}}$ root of the $m^{\text{th}}$ power of $x$; and $x^{\frac{mp}{np}}$ means the $\overline{np}|^{\text{th}}$ root of the $\overline{mp}|^{\text{th}}$ power of $x$, and these cannot be assumed to be the same until they are proved so.

vii. To show that $x^{\frac{p}{q}}.x^{\frac{r}{s}} = x^{\frac{p}{q}+\frac{r}{s}}$, where $p$, $q$, $r$, $s$, are positive integers.

For $\quad x^{\frac{p}{q}}.x^{\frac{r}{s}} = x^{\frac{ps}{qs}}.x^{\frac{qr}{qs}}$ by (vi.)

$$= (x^{ps})^{\frac{1}{qs}}.(x^{qr})^{\frac{1}{qs}} \text{ by (ii.)}$$

$$= (x^{ps}.x^{qr})^{\frac{1}{qs}} \text{ by (iv.)}$$

$$= (x^{ps+qr})^{\frac{1}{qs}}$$

$$= x^{\frac{ps+qr}{qs}}$$

$$= x^{\frac{p}{q}+\frac{r}{s}}.$$

Similarly it can be proved that—

$$x^{\frac{p}{q}} \div x^{\frac{r}{s}} = x^{\frac{p}{q}-\frac{r}{s}}.$$

Hence it appears that Rules (1) and (2), as applied to fractional indices, are a necessary consequence of the definition of a fractional index.

viii. Show that $\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}} = x^{\frac{pq}{rs}}.$

Begin by raising $\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}}$ to the $\overline{sq}|^{\text{th}}$ power.

**95.** Props. i. to viii. have been proved when $m$, $n$, $p$, $q$, $r$, $s$, are positive integers, or when the fractional indices are all positive; but they may all be extended to the cases where $m$, $n$, $p$, $q$, $r$, $s$ are any or all negative. Two examples will here be given of the mode of proof, when the indices are negative, and the other propositions should be worked out in a similar manner.

To prove $x^{\frac{1}{m}}.y^{\frac{1}{m}} = (xy)^{\frac{1}{m}}$, when $m$ is negative.

Since $m$ is negative it may be put $= -a \therefore \frac{1}{m} = -\frac{1}{a}$, where $a$ is a positive integer.

$$\text{Then} \qquad x^{\frac{1}{m}}.y^{\frac{1}{m}} = x^{-\frac{1}{a}}.y^{-\frac{1}{a}}$$

$$= \frac{1}{x^{\frac{1}{a}}}.\frac{1}{y^{\frac{1}{a}}} \text{ by Art. (93)}$$

$$= \frac{1}{(xy)^{\frac{1}{a}}} \text{ by (iv.)}$$

$$= (xy)^{-\frac{1}{a}}$$

$$= (xy)^{\frac{1}{m}}.$$

To prove $\left(x^{\frac{p}{q}}\right)^{\frac{m}{n}} = x^{\frac{pm}{qn}}$, when $m$ is negative, and $p$, $q$, $n$ positive.

Let $m = -a \therefore mp = -ap$.

$$\text{Now} \qquad \left(x^{\frac{p}{q}}\right)^{\frac{m}{n}} = \left(x^{\frac{p}{q}}\right)^{\left(-\frac{a}{n}\right)}$$

$$= \frac{1}{\left(x^{\frac{p}{q}}\right)^{\frac{a}{n}}}$$

$$= \frac{1}{x^{\frac{pa}{qn}}}$$

$$= x^{-\frac{ap}{qn}}$$

$$= x^{\frac{mp}{nq}}.$$

## Examples L.

1. Express the following in both notations:—

The square root of the 5th power of $a$.

*Ans.* $\sqrt{a^5}$, or $a^{\frac{5}{2}}$.

The $m^{th}$ root of the $p^{th}$ power of the product of the square of $a$, and the square root of $b$.

$$\textit{Ans.}\quad \sqrt[m]{(a^2\sqrt{b})^p},\ \text{or}\ \left(a^2b^{\frac{1}{2}}\right)^{\frac{p}{m}}.$$

The square root of $a$ divided by the cube root of $x$.

$$\textit{Ans.}\quad \frac{\sqrt{a}}{\sqrt[3]{x}},\ \text{or}\ \frac{a^{\frac{1}{2}}}{x^{\frac{1}{3}}},\ \text{or}\ a^{\frac{1}{2}}x^{-\frac{1}{3}}.$$

2. The fourth root of the square of $a$, the product of the square root of $x$, and the cube root of $y$. The fourth root of the product of the square root of the cube of $a$, and the cube root of the square of $b$. The fifth root of the square of the quotient of the cube root of the square of $a$, by the fourth root of the cube of $x$. The cube root of the $p^{th}$ power of the quotient of the $m^{th}$ power of $bx$, by the cube root of the $m^{th}$ power of $bx$.

3. Simplify $(a\sqrt{a})^{\frac{2}{3}}$.

Now $\qquad a\sqrt{a} = a.a^{\frac{1}{2}} = a^{\frac{3}{2}}$

And $\qquad (a\sqrt{a})^{\frac{2}{3}} = a^{\frac{3}{2}\cdot\frac{2}{3}} = a.$

Simplify $$\left(\frac{a^{\frac{m}{2}}}{a^{\frac{p}{3}}} \div \frac{a^{-\frac{m}{3}}}{a^{-\frac{p}{2}}}\right)^{\frac{6}{5}}$$

The expression in brackets $= a^{\frac{m}{2}-\frac{p}{3}} \div a^{\frac{p}{2}-\frac{m}{3}}$

$= a^{\frac{m}{2}-\frac{p}{3}} \times a^{-\frac{p}{2}+\frac{m}{3}}$

$= a^{\frac{5}{6}(m-p)}$

And $\qquad \left\{a^{\frac{5}{6}(m-p)}\right\}^{\frac{6}{5}} = a^{m-p}$, the result required.

4. Simplify $\left(\sqrt[3]{a^2}.\sqrt{a^3}\right)^6$; $\left(a^{\frac{1}{m}}.a^{\frac{1}{m}}\right)^{m+n}$; $\left(a^{\frac{1}{m}}.a^{\frac{1}{n}}a.^{\frac{1}{p}}\right)^{mnp}$;

$\dfrac{\left\{(x^p)^{\frac{1}{q}}.(x^m)^{\frac{1}{n}}\right\}^{qn}}{\left\{\sqrt[p]{y^q}.\sqrt[m]{y^n}\right\}^{pm}}$; $\sqrt[3]{a^4}.\sqrt[4]{a^5}.\sqrt[5]{a^6}$; $(49x^4y^3z^2)^{\frac{1}{2}}$;

$\dfrac{2a}{3b}\left(\dfrac{27}{8}a^3b^{-2}\right)^{\frac{1}{3}}$; $\dfrac{1}{x^2yz}(32x^{10}y^5z)^{-\frac{1}{5}}$; $\dfrac{1}{4}x^{\frac{2}{3}}.\left(x^{-\frac{3}{2}}\right)^{\frac{1}{3}}$;

$ax\,(ax)^{-\frac{1}{3}}$; $\left(\frac{a}{x}\right)^{\frac{2}{3}} a^{\frac{1}{2}} x^{-\frac{1}{2}}$; $\left(\frac{a}{x}\right)^{\frac{2}{3}} a^{\frac{1}{2}} x^{-\frac{1}{2}} \div \left(\frac{4a^2}{9x^2}\right)^{\frac{1}{4}}$;
$\left(\frac{a^3b^2}{64y^2}\right)^{-\frac{1}{6}} . \left(\frac{9x}{ay}\right)^{-\frac{1}{2}} . \left(\frac{bx}{y^2}\right)^{\frac{1}{3}}$; $(x^3y^p)^{\frac{1}{p}} \div (x^2y^{2p})^{\frac{1}{2p}}$;
$\left(\frac{4}{3}x^{\frac{3}{2}}\right)^{\frac{2}{3}} \div \frac{2}{3}\left(x^{-\frac{1}{2}}\right)^{-\frac{2}{3}}$; $(a^2b)^{\frac{1}{2}}.(ab^2)^{\frac{1}{3}}.(a^2b)^{-\frac{1}{2}}$
$.(a^2b^4)^{-\frac{1}{2}}$; $\left\{(x^{2p}.y^{-2q})^{\frac{m}{p}}\right\}^{\frac{n}{2q}}$; $\sqrt{\frac{a^nb^{2m}c^{m+3}}{x^{m-2}y^{m-3}}}$;
$\sqrt[2n]{\frac{a^{2np}.b^{2nq}}{x^{4n}.y^{3n}}}$.

5. Raise $x^{\frac{1}{2}}y^{-\frac{1}{3}}$ to the 6th power; $\sqrt[3]{x^2y^{-\frac{1}{2}}}$ to the 6th power; $-\frac{4\sqrt{a}}{3\sqrt[4]{b}}$ to the 4th power; $-a^{\frac{1}{3}}b^{-3}x^{-1}$ to the third power; $\sqrt{x\sqrt{x\sqrt{x}}}$ to the 8th power.

6. Find the square roots of the following quantities :—

$$a^{\frac{2}{3}}b^{\frac{3}{2}}c\,;\quad \frac{x^{\frac{2}{5}}y^{\frac{4}{5}}}{z^{\frac{5}{6}}}\,;\quad \frac{a^2.a^{\frac{1}{3}}}{(b^2)^{-\frac{1}{6}}}\,;\quad \frac{a^{\frac{6}{7}}}{b^{\frac{6}{11}}}\,;\quad \frac{a^{\frac{1}{6}}.b^{\frac{1}{10}}.c^{\frac{1}{2}}}{a^{\frac{1}{4}}b^{\frac{1}{8}}c^{\frac{1}{4}}}.$$

Also, find the cube root, fourth root, and fifth root of these quantities, and express the result without fractional or negative indices.

7. Multiply together the two quantities—

$$x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \text{ and } x^{\frac{1}{3}} - y^{\frac{1}{3}}.$$

The arrangement and process are the same as in multiplication with integral indices, and the work is given below.

$$\begin{array}{cccc}
x^{\frac{2}{3}} & + x^{\frac{1}{3}}y^{\frac{1}{3}} & + y^{\frac{2}{3}} & \\
 & x^{\frac{1}{3}} & - y^{\frac{1}{3}} & \\
\hline
x & + x^{\frac{2}{3}}y^{\frac{1}{3}} & + x^{\frac{1}{3}}y^{\frac{2}{3}} & \\
 & - x^{\frac{2}{3}}y^{\frac{1}{3}} & - x^{\frac{1}{3}}y^{\frac{2}{3}} & - y \\
\hline
x & * & * & - y \\
\hline
\end{array}$$

Find the products of the following factors—

$a^{\frac{1}{2}} + cd^{\frac{1}{3}}$ and $a^{\frac{1}{2}} - cd^{\frac{1}{3}}$; $a^{\frac{1}{2}} - a^{\frac{1}{3}} + a^{\frac{1}{4}}$ and $a^{\frac{1}{4}} - a^{\frac{1}{5}}$; $\sqrt{a^m} + \sqrt{b^m}$ and $\sqrt{a^m} - \sqrt{b^m}$; $a + 2b^{\frac{1}{2}} + 3c^{\frac{1}{3}}$ and $a - 2b^{\frac{1}{2}} + 3c^{\frac{1}{3}}$; $p^{\frac{1}{4}}q + q^{\frac{2}{3}}$ and $p^{\frac{1}{4}} - q^{\frac{1}{3}}$; $x^{\frac{5}{2}} + 3x^2y^{\frac{1}{3}} + 9x^{\frac{3}{2}}y^{\frac{2}{3}} + 27xy + 81x^{\frac{1}{2}}y^{\frac{4}{3}} + 243y^{\frac{5}{3}}$ by $x^{\frac{1}{2}} - 3y^{\frac{1}{3}}$; $(a + x)^{\frac{1}{n}}$, $(a + x)^{\frac{1}{m}}$, $(a - x)^{\frac{1}{m}}$ and $(a - x)^{\frac{1}{n}}$; $x^{\frac{2}{3}} + x^{-\frac{2}{3}} - \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right) + 1$ and $x^{\frac{1}{3}} + x^{-\frac{1}{3}} + 1$; $3x^{-4} - 2x^{-2}y + y^2$ and $x^{-2} - \frac{1}{2}x^{-1}y + \frac{1}{3}y^2$; $a^{3n} + a^{-3n}$ and $a^n - a^{-n}$; $\sqrt{abc}\,.\left(a^{\frac{1}{6}} + b^{\frac{1}{6}} + c^{\frac{1}{6}}\right)$ and $\sqrt[3]{a^2b^2c^2}\left(a^{\frac{1}{6}} - b^{\frac{1}{6}} + c^{\frac{1}{6}}\right)$.

8. Divide $x^2 - 1$ by $\sqrt{x} - 1$.

The arrangement and process will be the same as in division of integer indices, and is as follows:—

$$x^{\frac{1}{2}} - 1)\,x^2 - 1\,(x^{\frac{3}{2}} + x + x^{\frac{1}{2}} + 1$$

$$\begin{array}{l}
\underline{x^2 - x^{\frac{3}{2}}} \\
\qquad x^{\frac{3}{2}} - 1 \\
\qquad \underline{x^{\frac{3}{2}} - x} \\
\qquad\qquad x - 1 \\
\qquad\qquad \underline{x - x^{\frac{1}{2}}} \\
\qquad\qquad\qquad x^{\frac{1}{2}} - 1 \\
\qquad\qquad\qquad \underline{x^{\frac{1}{2}} - 1} \\
\qquad\qquad\qquad \underline{\quad . \quad .\quad}
\end{array}$$

This division might also have been performed by resolving the dividend into factors.

$$\text{Thus } x^2 - 1 = \left(x^{\frac{1}{2}}\right)^4 - 1^4$$

$$= \left(x^{\frac{1}{2}} - 1\right)\left(\left(x^{\frac{1}{2}}\right)^3 + \left(x^{\frac{1}{2}}\right)^2 . 1 + x^{\frac{1}{2}} . 1^2 + 1^3\right)$$

$$= \left(x^{\frac{1}{2}} - 1\right)\left(x^{\frac{3}{2}} + x + x^{\frac{1}{2}} + 1\right).$$

Most of the following examples may be done by breaking the dividend into factors by the methods of Chapter X.

Find the following quotients:—

$x^3 - 1$ by $x^{\frac{1}{3}} - 1$; $x^2 - x^{-2}$ by $x - x^{-1}$; $x^{\frac{1}{2}} - x^{-\frac{5}{2}}$ by $x^{\frac{3}{2}} - x^{-\frac{3}{2}}$; $x^{\frac{4}{3}} + x^{\frac{2}{3}} - 1$ by $x^{\frac{1}{3}} + 1 + x^{-\frac{1}{3}}$; $x^4 - 256y^4$ by $x^{\frac{1}{2}} - 2y^{\frac{1}{2}}$; $x^3 - y^3$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$; $a^{-1} - b^{-1}$ by $a^{-\frac{1}{4}} \pm b^{-\frac{1}{4}}$; $\dfrac{1}{a^3} - 64b^2$ by $2\sqrt[3]{b} + \dfrac{1}{\sqrt{a}}$; $x$ by $\sqrt[4]{x} - 1$ to 6 terms in the quotient; $x$ by $x^{\frac{1}{n}} + a^{\frac{1}{n}}$ to 6 terms; $(x^2 + y^2)(x - y)$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

In this example divide $x - y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$, and multiply the result by $x^2 + y^2$.

$x^5 + \dfrac{1}{x^5}$ by $\sqrt[5]{x} + \dfrac{1}{\sqrt[5]{x}}$; $x^{-1} + \frac{1}{8}$ by $\frac{1}{2} + x^{\frac{1}{3}}$ to 6 terms; $2a^2b - \dfrac{a}{\sqrt[3]{b}} + \dfrac{6}{a\sqrt[3]{b}} - \dfrac{3}{a^2b}$ by $2a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{-\frac{1}{2}}b^{-\frac{1}{2}}$; $\dfrac{a^2b^{\frac{3}{2}}}{729c^3} - \dfrac{64b^{\frac{9}{2}}c^4}{a^2}$ by $\frac{1}{3}a^{\frac{1}{3}}b^{\frac{1}{4}}c^{-\frac{1}{2}} - 2a^{-\frac{1}{3}}b^{\frac{3}{4}}c^{\frac{2}{3}}$; $x + y^2 + z^3 - 3(xy^2z^3)^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{2}{3}} + z$.

9. Extract the square root of $4a - 12a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{1}{3}} + 9b^{-1}c^{-\frac{2}{3}}$.

$$\begin{array}{l|l}
 & 4a - 12a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{1}{3}} + 9b^{-1}c^{-\frac{2}{3}}\big(2a^{\frac{1}{2}} - 3b^{-\frac{1}{2}} \\
 & \underline{4a\qquad\qquad\qquad\qquad\qquad\qquad} \quad +c^{-\frac{1}{3}}. \\
4a^{\frac{1}{2}} - 3b^{-\frac{1}{2}} & -12a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{1}{3}} + 9b^{-1}c^{-\frac{2}{3}} \\
+c^{-\frac{1}{3}} & \underline{-12a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{-\frac{1}{3}} + 9b^{-1}c^{-\frac{2}{3}}} \\
 & \overline{\qquad\qquad\qquad\qquad\qquad}
\end{array}$$

From this working, it is seen that the arrangement and process are the same as in quantities with integral indices.

Find the following square roots:—

$$9a^3 - 6a^{\frac{3}{2}}b^{\frac{1}{2}} + b;\ \tfrac{1}{4}a^4 - 2a^2b^{\frac{1}{2}} - 4b;\ \frac{4a^{2m}}{b^{2n}} - 3 + \frac{9b^{2n}}{16a^{2m}};\ \frac{a^4b}{4} - a^3 + \tfrac{3}{2}\frac{a^2}{b} + \frac{a}{b^2} + \frac{1}{4b^3};\ \frac{4}{25}a^{bm}.b^{\frac{2}{m}} - 4a^{2m}b^{\frac{3}{m}} + 25a^{-2m}b^{\frac{4}{m}};\ x^{2+\frac{1}{m}}\left(x^{\frac{1}{m}-2}y^{2m} - 1\right) + \frac{x^4}{4y^{2m}};\ 3x^{\frac{2}{p}} + 2 + x^{-\frac{2}{p}} + x^{\frac{4}{p}} + x^{\frac{6}{p}}.$$

10. Extract the cube roots of the following quantities:—

$$x^3 + x^{-3} + 3(x - x^{-1});\ \frac{1}{8a^6} + \frac{3}{4a^4} + \frac{3}{2a^2} + 1;\ \frac{x^6y^{\frac{3}{2}}}{8} - \frac{x^5y^{\frac{1}{2}}}{4} + \frac{x^4}{6y^{\frac{1}{2}}} - \frac{x^3}{27y^{\frac{3}{2}}};\ 8a^{3m} + 6a^{\frac{3m}{2}} + \tfrac{3}{2} + \tfrac{1}{8}a^{-\frac{3m}{2}};\ x^6 - 3x^{\frac{16}{3}} + 3x^{\frac{14}{3}} + 3x^5 - 6x^{\frac{13}{3}} + 2x^4 + 3x^{\frac{11}{3}} + x^3;\ x^{\frac{1}{3}}\left(x^{\frac{1}{3}} + a^{\frac{1}{3}}\right)^2 + a^{\frac{1}{3}}\left(a^{\frac{1}{3}} + x^{\frac{1}{3}}\right)^2;$$

$$\left(125x^{-1} - 75x^{-\frac{7}{4}} + 15x^{-\frac{5}{2}} - x^{-\frac{13}{3}}\right)y^{\frac{2}{5}};\ \left(x^{\frac{1}{2}} + y^{\frac{2}{3}}\right)^3.\left(x^{\frac{2}{3}} - y^{-\frac{1}{2}}\right)^{-3}\left(x^{\frac{1}{4}}y^{\frac{1}{4}}\right)^{-9};\ \left(27x^6y^{-9} + 108x^3y^{-4} + 144y + 64x^{-3}y^6\right)^{-9};$$

$$\left\{\frac{x}{y} + \frac{y}{x} + 3\left(\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} + \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)\right\}^{-2} \div \left\{\frac{x}{y} - \frac{y}{x} + 3\left(\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} + \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}\right)\right\}^{\frac{1}{3}}.$$

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## SURDS.

**96.** When a proposed root of an algebraical quantity n be found, that root is called a Rational Quantity; but the root cannot be found, the expression denoting the ot is called a *Surd*, or Irrational Quantity.

Thus $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is a surd.

But $\sqrt[3]{8^2}$ or $8^{\frac{2}{3}}$ is a rational quantity,

For $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4$.

So $\sqrt[3]{5}$, $\sqrt{a^2 + x^2}$, are surds.

**97.** Some examples will now be given of the most usual operations necessary to be performed on surd quantities.

*Any power of a quantity may be denoted by any index, whose value is the same as that of the given index.*

Thus $x = x^{\frac{2}{2}} = x^{\frac{3}{3}} = x^{\frac{n}{n}}$, &c.

Or, $x = \sqrt{x^2} = \sqrt[3]{x^3} = \sqrt[n]{x^n}$, &c.

And $x^{\frac{2}{3}} = x^{\frac{4}{6}} = x^{\frac{6}{9}}$, &c.

$$(a + x)^{\frac{1}{n}} = (a + x)^{\frac{2}{2n}} = (a + x)^{\frac{m}{mn}}, \text{ \&c.}$$

Hence the product of any number of different quantities may be expressed under the same radical sign.

Thus $a^2 b^{\frac{1}{2}} c^{\frac{2}{3}} = a^{\frac{12}{6}} b^{\frac{3}{6}} c^{\frac{4}{6}} = (a^{12} b^3 c^4)^{\frac{1}{6}} = \sqrt[6]{a^{12} b^3 c^4}$.

Conversely, the rational factors contained in a surd may be removed from under the radical sign.

Thus
$$\begin{aligned}\sqrt[3]{a^4 b^5 c^6} &= (a^4 b^5 c^6)^{\frac{1}{3}} = a^{\frac{4}{3}} b^{\frac{5}{3}} c^{\frac{6}{3}} \\ &= a . a^{\frac{1}{3}} b . b^{\frac{2}{3}} . c^2 \\ &= abc^2 a^{\frac{1}{3}} . b^{\frac{2}{3}} \\ &= abc^2 (ab^2)^{\frac{1}{3}}.\end{aligned}$$

And $\sqrt[m]{x^m . y^{2m} . z^{3m} (x + y)^{m+1} . (y + z)^{m+2} . (x + z)^{m+3}}$

$$= x . y^2 z^3 (x + y)(x + y)^{\frac{1}{m}} . (y + z)(y + z)^{\frac{2}{m}} (x + z) (x + z)^{\frac{3}{m}}$$

$$= xy^2 z^3 (x + y)(y + z)(x + z) \{(x + y)(y + z)^2 (x + z)^3\}^{\frac{1}{m}}.$$

**98.** Any quantity may be made the co-efficient of a given surd, if every term under the sign be divided by that quantity raised to the power whose root the sign expresses.

Ex. To make $a$ the coefficient of the surd $(a^4 + a^2x^2)^{\frac{1}{2}}$.

The result is $a\,(a^2 + x^2)^{\frac{1}{2}}$.

$$\text{For}\quad (a^4 + a^2x^2)^{\frac{1}{2}} = \frac{a\,(a^4 + a^2x^2)^{\frac{1}{2}}}{a} = a\frac{(a^4 + a^2x^2)^{\frac{1}{2}}}{(a^2)^{\frac{1}{2}}}$$

$$= a\left(\frac{a^4 + a^2x^2}{a^2}\right)^{\frac{1}{2}}$$

$$= a\,(a^2 + x^2)^{\frac{1}{2}}.$$

Similarly, if it be required to make $a^2$ the co-efficient of the surd—

$$(a^{2m} + a^m x^m + x^{2m})^{\frac{1}{n}}$$

It takes the form—

$$a^2\,(a^{2m-2n} + a^{m-2n}\,x^m + a^{-2n}\,x^{2m})^{\frac{1}{n}}.$$

So also—

$$\sqrt[3]{8a^2 + 8ax + 2x^2} = 2\left(a^2 + ax + \frac{x^2}{4}\right)^{\frac{1}{3}}.$$

**99.** Surds of more than one term may be bracketed, when the same surd factor is common to them all.

$$\text{Thus}\quad ab\sqrt{x} + bc\sqrt{x} + ac\sqrt{x}$$

$$= \sqrt{x}\,(ab + bc + ac).$$

$$\text{Also}\quad \sqrt{9a^3bc} - \sqrt{16ab^3c} + \sqrt{4abc^3}$$

$$= (abc)^{\frac{1}{2}}\left\{(9a^2)^{\frac{1}{2}} - (16b^2)^{\frac{1}{2}} + (4c^2)^{\frac{1}{2}}\right\}$$

$$= (abc)^{\frac{1}{2}}\,\{3a - 4b + 2c\}.$$

**100.** When two or more surds have the same index, their product may be found by multiplying together the quantities under the radical sign, and writing that product under the same index.

$$\text{Thus,}\quad (a + 1)^{\frac{1}{2}}(a - 1)^{\frac{1}{2}}(a^2 + 1)^{\frac{1}{2}} = (a^4 - 1)^{\frac{1}{2}}$$

And $2a^{\frac{1}{3}}.3b^{\frac{1}{3}}.a(a+b)^{\frac{1}{3}}.b(a-b)^{\frac{1}{3}}$

$$= 6ab\,\{ab(a^2-b^2)\}^{\frac{1}{3}}.$$

Sometimes it is necessary to reduce the surds to indices with a common denominator.

Thus $(a+x)^{\frac{1}{2}}(a-x)^{\frac{1}{4}}(ax)^{\frac{1}{2}}$

$$= (a+x)^{\frac{2}{4}}(a-x)^{\frac{1}{4}}(ax)^{\frac{2}{4}}$$

$$= (a^2+2ax+x^2)^{\frac{1}{4}}(a-x)^{\frac{1}{4}}(a^2x^2)^{\frac{1}{4}}$$

$$= (a^5x^2+a^4x^3-a^3x^4-a^2x^5)^{\frac{1}{4}}.$$

**101.** Different powers of the same surd are multiplied together by adding the indices of all the factors, and divided by subtracting the index of the divisor from that of the dividend.

Ex. Divide $(ax+by)^{\frac{2}{3}}$ by $(ax+by)^{\frac{1}{2}}$ .

The quotient $= (ax+by)^{\frac{2}{3}-\frac{1}{2}}$

$$= (ax+by)^{\frac{1}{6}}.$$

Multiply together $4^{\frac{2}{3}}$, $2^{\frac{3}{2}}$, and $8^{\frac{1}{6}}$.

These different surds may all be expressed as powers of the same surd, for

$$4^{\frac{2}{3}} = (2^2)^{\frac{2}{3}} = 2^{\frac{4}{3}}$$

And $8^{\frac{1}{6}} = (2^3)^{\frac{1}{6}} = 2^{\frac{3}{6}}$

$\therefore$ the product $= 2^{\frac{4}{3}+\frac{3}{2}+\frac{3}{6}}$

$$= 2^{\frac{10}{3}}$$

$$= 8 \times 2^{\frac{1}{3}}.$$

Divide $(27)^{\frac{4}{3}}$ by $(9)^{\frac{5}{4}}$

$$(27)^{\frac{4}{3}} = (3^3)^{\frac{4}{3}} = 3^4$$

$$(9)^{\frac{5}{4}} = (3^2)^{\frac{5}{4}} = 3^{\frac{5}{2}}.$$

And $3^4 \div 3^{\frac{5}{2}} = 3^{\frac{3}{2}} = \sqrt{27}.$

**102.** When the denominator of a fraction contains surds, it may be transformed into an equivalent fraction without surds in its denominator. This process is called "Rationalizing the Denominator."

Thus
$$\frac{a^2b^{\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{1}{3}}} = \frac{a^2b^{\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{1}{3}}} \times \frac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$$
$$= \frac{a^2b^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{3}}}{xy}.$$

Also
$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

*Note.*—Observe that the factor introduced into the numerator and denominator must be a surd, whose index added to the index of the denominator makes an integer.

When the denominator consists of *two* terms, one or each of which is a Quadratic surd, a different method must be used.

Ex.
$$\frac{3}{\sqrt{2}+1} = \frac{3}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$
$$= \frac{3\sqrt{2}-3}{2-1}$$
$$= 3\sqrt{2}-3$$

Again,
$$\frac{2\sqrt{a}+\sqrt{3b}}{3\sqrt{a}-\sqrt{5b}} = \frac{2\sqrt{a}+\sqrt{3b}}{3\sqrt{a}-\sqrt{5b}} \times \frac{3\sqrt{a}+\sqrt{5b}}{3\sqrt{a}+\sqrt{5b}}$$
$$= \frac{6a+(2\sqrt{5}+3\sqrt{3})\sqrt{ab}+b\sqrt{15}}{9a-5b}$$

Also
$$\frac{3\sqrt{5}}{\sqrt{5}+\sqrt{2}} = \frac{3\sqrt{5}(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})} = \frac{15-3\sqrt{10}}{5-2}$$
$$= \frac{5-\sqrt{10}}{1}$$

This transformation of fractions is useful when their numerical values are required, correct to a given number of decimal places. For instance, let the numerical value of the fraction $\frac{3\sqrt{5}}{\sqrt{5}+\sqrt{2}}$ be required correct to 4 places of decimals.

Taken in its present form, the calculation would be as follows:—

1. Extract the square root of 5 to 5 places of decimals, and multiply it by 3 for the value of the numerator.
2. Extract the square root of 2 to 5 places of decimals, and add it to the value already found of $\sqrt{5}$ for the denominator.
3. Divide the value of the numerator by that of the denominator by long division to 4 places of decimals.

But if we take it in its second form $5 - \sqrt{10}$, the calculation is effected by extracting the square root of 10 to 5 places of decimals and subtracting it from 5.

If the pupil will actually work out this example by the two different methods, he will be convinced of the advantage of the latter.

The same remarks apply to the fraction $\frac{3}{\sqrt{5}}$.

The Rule for rationalizing the denominator of a fraction when it consists of *two* terms, one or each of which is a Quadratic surd, may be thus stated:—

### RULE.

*If the Denominator be the sum of the two terms, multiply Numerator and Denominator by the difference of these terms; but if the Denominator be the difference of the two terms, multiply by the* sum.

Thus, to rationalize the denominator of the fraction $\frac{1}{\sqrt{a}+\sqrt{b}}$, we use the multiplier $\sqrt{a}-\sqrt{b}$.

But in the fraction

$\frac{1}{\sqrt{a}-\sqrt{b}}$, we use the multiplier $\sqrt{a}+\sqrt{b}$.

*Note.*—This method applies only to *Quadratic* surds, that is, surds in which the square root is denoted.

**103.** When the denominator of the fraction is a Binomial, and the surd or surds are not Quadratic, a different method must be adopted, which will now be explained.

The fraction will be of one of the two forms—

$$\frac{c}{a^{\frac{1}{m}}-b^{\frac{1}{n}}} \text{ or } \frac{c}{a^{\frac{1}{m}}+b^{\frac{1}{n}}},$$

where $c$ denotes any algebraical expression which stands in the numerator.

i. Consider the fraction $\frac{c}{a^{\frac{1}{m}}-b^{\frac{1}{n}}}$.

$$\text{For simplicity, put } \left.\begin{aligned} a^{\frac{1}{m}} &= x \\ \text{and } b^{\frac{1}{n}} &= y \end{aligned}\right\}$$

$$\begin{aligned} \therefore \quad a &= x^m \\ \text{and} \quad b &= y^n. \end{aligned}$$

Let $l$ be the L. C. M. of $m$ and $n$.

$$\begin{aligned} \text{Then} \quad \left(a^{\frac{1}{m}}\right)^l &= x^l \\ \text{and} \quad \left(b^{\frac{1}{n}}\right)^l &= y^l \end{aligned}$$

and $x^l$ and $y^l$ are both rational.

$$\text{Now, } x^l - y^l = (x-y)(x^{l-1} + x^{l-2}y + x^{l-3}y^2 + \ldots + x^2y^{l-3} + xy^{l-2} + y^{l-1}).$$

see p. 29. (1).

If, then, the numerator and denominator of the fraction $\frac{c}{x-y}$ be multiplied by the expression within the brackets $(x^{l-1} + x^{l-2}y +$ , &c.) it becomes

$$\frac{c\,(x^{l-1} + x^{l-2}y + \ldots\ldots\ldots + xy^{l-2} + y^{l-1})}{x^l - y^l}$$

a fraction with a rational denominator.

And if instead of $x$ we write $a^{\frac{1}{m}}$, and instead of $y$ we write $b^{\frac{1}{n}}$, the fraction will be in the form required.

ii. When the fraction is of the form $\dfrac{c}{a^{\frac{1}{m}} + b^{\frac{1}{n}}}$, two cases will arise according as $l$, the L. C. M. of $m$ and $n$, is even or odd.

Take $x$, $y$, $l$ as before.

Then, if $l$ be even,

$$x^l - y^l = (x + y)\,(x^{l-1} - x^{l-2}y + x^{l-3}y^2 - \ldots\ldots \\ + xy^{l-2} - y^{l-1}). \qquad (2)$$

But if $l$ be odd,

$$x^l + y^l = (x + y)\,(x^{l-1} - x^{l-2}y + x^{l-3}y^2 - \ldots\ldots \\ - xy^{l-2} + y^{l-1}). \qquad (3)$$

Hence we see that if $l$ be even, the rational denominator will be $x^l - y^l$, and the multiplier will be the quantity within brackets in (2): but if $l$ be odd, the rational denominator will be $x^l + y^l$, and the multiplier will be the quantity within brackets (3); $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ being substituted for $x$ and $y$ as before.

Ex. 1. Let $\dfrac{1}{a^{\frac{1}{3}} - b^{\frac{1}{2}}}$ be the fraction to be rationalized.

Here $a^{\frac{1}{3}} = x$

$b^{\frac{1}{2}} = y$

$l = 6.$

Now $$x^6 - y^6 = (x - y)\,(x^5 + x^4y + x^3y^2 + x^2y^3 \\ + xy^4 + y^5)$$

Or, $$a^2 - b^3 = \left(a^{\frac{1}{3}} - b^{\frac{1}{2}}\right)\left(a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{2}} + ab + a^{\frac{2}{3}}b^{\frac{3}{2}} \\ + a^{\frac{1}{3}}b^2 + b^{\frac{5}{2}}\right)$$

$$\therefore\ \frac{1}{a^{\frac{1}{3}} - b^{\frac{1}{2}}} = \frac{a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{2}} + ab + a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 + b^{\frac{5}{2}}}{a^2 - b^3}.$$

Ex. 2. Let $\dfrac{1}{a^{\frac{1}{3}} + b^{\frac{1}{2}}}$ be the fraction.

Here $x$, $y$, $l$ are the same as in Ex. 1.

And since $l$ is even, we have

$$x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5)$$

and making the necessary substitutions for $x$ and $y$, we have

$$\frac{1}{a^{\frac{1}{3}} + b^{\frac{1}{2}}} = \frac{a^{\frac{5}{3}} - a^{\frac{4}{3}}b^{\frac{1}{2}} + ab - a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 - b^{\frac{5}{2}}}{a^2 + b^3}$$

the fraction required.

Ex. 3. Let $\dfrac{a^{\frac{1}{3}} - b^{\frac{1}{5}}}{a^{\frac{1}{3}} + b^{\frac{1}{5}}}$ be the fraction.

Here $l = 15$, $x = a^{\frac{1}{3}}$, $y = b^{\frac{1}{5}}$.

And $x^{15} + y^{15} = (x + y)(x^{14} - x^{13}y + \ldots\ldots\ldots\ldots - xy^{13} + y^{14})$,

$$\therefore \frac{a^{\frac{1}{3}} - b^{\frac{1}{5}}}{a^{\frac{1}{3}} + b^{\frac{1}{5}}} \times \frac{\left(a^{\frac{14}{3}} - a^{\frac{13}{3}}y + \ldots\ldots - a^{\frac{1}{3}}y^{\frac{13}{5}} + y^{\frac{14}{5}}\right)}{\left(a^{\frac{14}{3}} - a^{\frac{13}{3}}y + \ldots\ldots - a^{\frac{1}{3}}y^{\frac{13}{5}} + y^{\frac{14}{5}}\right)}$$

$$= \frac{a^5 - 2a^{\frac{14}{3}}b^3 + \&c.\ldots\ldots}{a^5 + b^3}.$$

The whole of the multiplication indicated in the numerator should be worked out.

Ex. 4. $\dfrac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$.

Here the multiplier is $\sqrt{5} + \sqrt{3}$, and the fraction becomes—

$$\frac{8 + 2\sqrt{15}}{2} = 4 + \sqrt{15}.$$

**104.** If the denominator contain *three* quadratic surds, it may be rationalized by repeating the above process, as in the following example—

$$\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{7}}.$$

Multiply first by $\sqrt{2} + \sqrt{3} - \sqrt{7}$, the fraction becomes—

$$\frac{\sqrt{2}+\sqrt{3}-\sqrt{7}}{-2+2\sqrt{6}}=\frac{\sqrt{2}+\sqrt{3}-\sqrt{7}}{2(\sqrt{6}-1)}.$$

Multiply secondly by $\sqrt{6}+1$, then it becomes—

$\dfrac{(\sqrt{2}+\sqrt{3}-\sqrt{7})(\sqrt{6}+1)}{2\times 5}$, which may be worked out.

**105.** The results of the two following propositions will be found useful in extracting the roots of quantities involving quadratic surds.

i. The square root of a rational quantity cannot be partly rational and partly a quadratic surd.

Let $a$ be the rational quantity, and, if possible, let

$$\sqrt{a}=x+\sqrt{y}.$$

Squaring both sides of this equation, we have

$$a=x^2+2x\sqrt{y}+y$$

$$\therefore\quad 2x\sqrt{y}=a-x^2-y$$

Or, $$\sqrt{y}=\frac{a-x^2-y}{2x}.$$

That is, a quadratic surd is equal to a rational quantity, which is absurd; hence the square root cannot be of the form supposed.

ii. If the two sides of an equation contain both rational quantities and quadratic surds, the rational quantities on each side are equal to each other, as also the surds.

Let $x+\sqrt{y}=a+\sqrt{b}$ be the equation, where $x$ and $a$ denote the sum of the rational quantities on each side, and $\sqrt{b}$, $\sqrt{y}$ the sum of the quadratic surds on each side.

Then shall $$x=a$$

And $$\sqrt{y}=\sqrt{b},$$

For if $x$ be not equal to $a$,

Let $$x=a+h.$$

Then the original equation becomes

$$a+h+\sqrt{y}=a+\sqrt{b}$$

$$\therefore \qquad h + \sqrt{y} = \sqrt{b},$$

which is impossible by the first part; whence we must have

$$x = a$$

$$\therefore \qquad \sqrt{y} = \sqrt{b}.$$

**106.** By the help of the preceding propositions, we may now investigate a method of finding the square root of a binomial, which has one of its terms rational and the other a quadratic surd.

Let $a + \sqrt{b}$ be the given binomial.

Assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$

Square both sides.

Then
$$a + \sqrt{b} = x + 2\sqrt{xy} + y$$

$$\therefore \qquad a = x + y$$

And
$$\sqrt{b} = 2\sqrt{xy} \quad \text{by (105). ii.}$$

Squaring each of the last equations, we get

$$x^2 + 2xy + y^2 = a^2$$

$$4xy = b.$$

By subtraction $x^2 - 2xy + y^2 = a^2 - b$

Extracting square root $x - y = \sqrt{a^2 - b}.$

If this root can be extracted, call it $c$.

$$\therefore \qquad x - y = c$$

But
$$x + y = a.$$

By addition of the last two equations

$$2x = a + c$$

Or,
$$x = \frac{a + c}{2}$$

By subtraction
$$2y = a - c$$

Or,
$$y = \frac{a - c}{2}$$

And then $\sqrt{x} + \sqrt{y}$ is knwon.

*Note.* If the square root of $(a^2 - b)$ cannot be extracted, the process is useless, since the expressions for $x$ and $y$ will be too complicated.

The chief use of this method is to extract the square roots of numerical quantities, as in the following example.

Find the square root of $9+4\sqrt{5}$.

Let $$\sqrt{9+4\sqrt{5}} = \sqrt{x}+\sqrt{y}$$

Squaring, we have $$9+4\sqrt{5} = x+2\sqrt{xy}+y.$$

$$\therefore \quad x+y = 9$$
$$2\sqrt{xy} = 4\sqrt{5},$$

and proceeding as in the proposition above, we shall find

$$x+y = 9$$

And $$x-y = 1$$

Whence $$x = 5$$
$$y = 4$$

$$\therefore \quad \sqrt{9+4\sqrt{5}} = \sqrt{5}+2.$$

If it had been required to find $\sqrt{9-4\sqrt{5}}$, we should have assumed

$$\sqrt{9-4\sqrt{5}} = \sqrt{x}-\sqrt{y}$$

and the process is similar to that just given, and the result is $$\sqrt{9-4\sqrt{5}} = \sqrt{5}-2.$$

*Note.* The labour of the preceding process may often be avoided, and the square root of the binomial found by inspection. For instance, if $4+2\sqrt{3}$ be the quantity whose square root is required, it can be put in the form $1^2+2\sqrt{3}+(\sqrt{3})^2$, which is evidently the square of $1+\sqrt{3}$, for it is made up of the squares of the terms 1 and $\sqrt{3}$, together with twice their product.

In the examples (LI.) of finding the square roots of binomials, both methods should be tried.

**107.** It is sometimes possible to find the square root of a quantity which consists of a rational term and three quadratic surds.

Let $m+\sqrt{a}+\sqrt{b}+\sqrt{c}$ be the quantity.

Assume $$\sqrt{m+\sqrt{a}+\sqrt{b}+\sqrt{c}} = \sqrt{x}+\sqrt{y}+\sqrt{z}.$$

Square both sides of this equation, we get

$$m+\sqrt{a}+\sqrt{b}+\sqrt{c} = x+y+z+2\sqrt{xy}+2\sqrt{xz} + 2\sqrt{yz}.$$

Put $2\sqrt{xy} = \sqrt{c},\ 2\sqrt{xz} = \sqrt{b},\ 2\sqrt{yz} = \sqrt{c}$

Multiplying these three equations together, we get

$$8xyz = \sqrt{abc}.$$

And by squaring the equation $2\sqrt{y2} = \sqrt{c}$, we get

$$4yz = c$$

Hence, by division $$2x = \frac{\sqrt{abc}}{c}$$

$$\therefore \quad x = \tfrac{1}{2}\sqrt{\frac{ab}{c}}$$

Similarly the values of $y$ and $z$ may be found.

It will now be necessary to examine whether the values of $xyz$ thus found, will satisfy the equation

$$x + y + z = m.$$

If this be the case, $\sqrt{x} + \sqrt{y} + \sqrt{z}$ will be the true square root, when the values of each letter are substituted; bnt if not, the square root cannot be found by this method.

Ex. Find the square root of $6 + 2\sqrt{3} - 2\sqrt{2} - 2\sqrt{6}$

Then $$\sqrt{6 + 2\sqrt{3} - 2\sqrt{2} - 2\sqrt{6}} = \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

By the same method as above, we have

$$2\sqrt{xy} = -2\sqrt{6}$$
$$2\sqrt{xz} = -2\sqrt{2}$$
$$2\sqrt{yz} = 2\sqrt{3}.$$

And hence $$x = 2$$
$$y = 3$$
$$z = 1.$$

And we see that $x + y + 2 = 6.$

And the required square root is

$$\sqrt{3} - \sqrt{2} + 1.$$

Respecting the negative sign in $-\sqrt{2}$, see Art. (110).

**108.** To find the cube root of a binomial of the form

$$a \pm \sqrt{b}$$

Assume $$\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}. \qquad (1.)$$

Cube both sides.

Then $a + \sqrt{b} = x^3 + 3x^2y^{\frac{1}{2}} + 3xy + y^{\frac{3}{2}}$

Hence $x^3 + 3xy = a$

And $3x^2y^{\frac{1}{2}} + y^{\frac{3}{2}} = \sqrt{b}.$ Art. (105.)

By subtracting one of these equations from the other, we get

$$x^3 - 3x^2y^{\frac{1}{2}} + 3xy - y^{\frac{3}{2}} = a - \sqrt{b},$$

Or, $x - \sqrt{y} = \sqrt[3]{a - \sqrt{b}}.$ (2.)

Multiply (1) and (2) together, we find

$$x^2 - y = \sqrt[3]{a^2 - b}.$$

If $a^2 - b$ be a perfect cube, its root can be extracted; call it $c$,

Then $x^2 - y = c,$ (3.)

But $x^3 + 3xy = a,$ (4.)

Multiply (3) by $3x$, $\therefore\ 3x^3 - 3xy = 3cx.$

Add the two last, $\therefore\ 4x^3 = 3cx + a,$

Or, $4x^3 - 3cx = a.$

From this equation $x$ must be found by *trial*, since no regular method has been given for finding the root of an equation involving $x^3$; and if a value of $x$ can be found in this way, $y$ may be found from (3), and thus the cube root will be known.

It is evident that if $a^2 - b$ be not a perfect cube, the cube root cannot be obtained by this method.

Ex. Find the cube root of $100 - 51\sqrt{3}.$

If we proceed as in the proposition,

Then $\sqrt[3]{100 - 51\sqrt{3}} = x - \sqrt{y}$

$\therefore\ 100 - 51\sqrt{3} = x^3 - 3x^2\sqrt{y} + 3xy - \sqrt{y^3}$

Whence $x^3 + 3xy = 100$

And $3x^2y^{\frac{1}{2}} + y^{\frac{3}{2}} = 51\sqrt{3}$

Also $\sqrt[3]{100 + 51\sqrt{3}} = x + \sqrt{y}$

$$\therefore\ x^2 - y = \sqrt[3]{10000 - 7803}$$

$$= \sqrt[3]{2197}$$

Or, $x^2 - y = 13$

$$\begin{aligned} &\text{But} & x^3 + 3xy &= 100 \\ &\therefore & 4x^3 - 39x &= 100. \\ &\text{We find by trial that} & x &= 4 \\ &\text{Also} & y &= x^2 - 13 \\ &\text{Or,} & y &= 3 \\ &\text{Hence} & \sqrt[3]{100 - 51\sqrt{3}} &= 4 - \sqrt{3}. \end{aligned}$$

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## Examples LI.

1. Express $ab^{\frac{1}{2}}c^{\frac{1}{4}}$ as a square root.

$$\begin{aligned} \text{Here} \qquad ab^{\frac{1}{2}}c^{\frac{1}{4}} &= a^{\frac{2}{2}}b^{\frac{1}{2}}(c^{\frac{1}{2}})^{\frac{1}{2}} \\ &= (a^2bc^{\frac{1}{2}})^{\frac{1}{2}} \\ &= \sqrt{a^2bc^{\frac{1}{2}}}. \end{aligned}$$

The process consists in reducing each of the indices to equivalent fractions having 2 for their denominator. If it be required to express the same quantity as a cube root, each of the indices must be reduced to equivalent fractions having 3 for the denominator, and so for the other roots.

2. Express $ab^{\frac{1}{2}}c^{\frac{1}{4}}$ as an $m^{\text{th}}$ power.

$$\begin{aligned} \text{Here} \qquad ab^{\frac{1}{2}}c^{\frac{1}{4}} &= a^{\frac{m}{m}}.b^{\frac{m}{2m}}.c^{\frac{m}{4m}} \\ &= \left(a^{\frac{1}{m}}.b^{\frac{1}{2m}}.c^{\frac{1}{4m}}\right)^m. \end{aligned}$$

In this case the index of each letter must be reduced to an equivalent fraction having $m$ for its numerator.

3. Express $4x^2y^3z^4$ as a cube; $\dfrac{m^2}{n^2}$ as a square; $\dfrac{m^{\frac{1}{2}}}{n^{\frac{1}{3}}}$ as a fourth power; $(a+b)^{\frac{2}{3}}$ as a cube root; $(ax+by)^{\frac{1}{3}}$ as a square root; $\sqrt{\dfrac{ay}{x}}$ as a fifth power; $2(4a^2b)^{\frac{1}{3}}$ and $ab^5c^{\frac{1}{2}}$ as fifth roots; $\left(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}\right)^{10}$ as a cube root of a fifth power.

4. Express the following as $m^{\text{th}}$ powers and $m^{\text{th}}$ roots.

$a^{2m}b^{3m}$; $a^2b^3$; $x^2y^{-2}$; $3a^6b^{2m}c^{m+3}$; $\sqrt[3]{a^2+x^2}$;

$(m^2 - n^2)^{\frac{m}{2}}$; $(abc)^{mn}$; $(ab^{-1})^{\frac{m}{n}}$; $(x + y + z)^{3m+1}$; $(x^2y^2z^2)^{-\frac{m}{2}}$.

5. Express $\left\{ab^2(ab^5)^{\frac{1}{2}}\right\}^{\frac{1}{5}}$ as a cube root and as a cube.

Also $\left(- 3x^2y^{\frac{5}{2}}z^{\frac{1}{3}}\right)$ as a square root and as an $m^{\text{th}}$ root.

$\dfrac{7a^2a^{\frac{1}{3}}}{(3b^2)^{\frac{1}{6}}}$ as a twelfth root.

6. Express with indices, whose numerator is 1, the following quantities—

$$4^{\frac{3}{4}};\ 2^{-\frac{3}{4}};\ 3^{\frac{2}{3}};\ 9^{-\frac{2}{5}};\ \left(\frac{2}{3}\right)^{-\frac{1}{2}};\ \left(\frac{1}{2}\right)^{-\frac{2}{5}}.$$

7. Reduce $a^nb^{\frac{1}{2}}c^{\frac{1}{3}}$ to the form of a $3m^{\text{th}}$ root.

Also $a^{\frac{1}{2}}bc^{\overline{n+1}}$ to the form of an $n^{\text{th}}$ root; also, $x^ny^{\frac{p}{m}}z^q$ as an $\overline{m+1}^{\text{th}}$ root; and $a^{\frac{p}{q}}b^qx^3y^q$ as a $\left(\frac{p}{q}\right)^{\text{th}}$ root; $(a + b + c)^2$ as a cube root; $(a + b)^2$ as a fourth power; $\sqrt{a^2 + b^2}$ as a cube root; $(a + x)^{\frac{m}{n}}$ as a $2m^{\text{th}}$ power; $\left(\dfrac{x - y}{x + y}\right)^{\frac{1}{3}}$ as a $3p^{\text{th}}$ root.

8. Take out the rational factors from the following surds :—

$$(27a^7b^6)^{\frac{1}{3}};\ \left(\frac{x^4z}{y^3}\right)^{\frac{1}{3}};\ \left(\frac{16a^5y^9}{9}\right)^{\frac{1}{2}};\ \left(\frac{a^8b^4}{c^5}\right)^{-\frac{1}{4}};\ \sqrt[3]{48};$$

$$\sqrt[3]{128x^9y^2};\ \sqrt{72a^2b^3};\ \sqrt{\frac{50}{147}};\ \sqrt{100a^4b^5};$$

$$\frac{3}{4}\sqrt{\frac{12}{75}};\ \sqrt[2n]{\frac{a^{2mn}b^{2n+1}}{x^{4n-3}y^{4mn}}};\ \sqrt[3]{27a^{3m-1}b^6};\ \sqrt[n]{x^{3n}y^{4n}z^5};$$

$$\sqrt{\frac{a^4 - a^3x}{a^3x - a^2x^2}};\ (x + y)^3 + (x^2 - y^2)^{\frac{3}{2}};$$

$$(x + y)^{\frac{3}{2}} + \sqrt{(x - y)^2(x + y)};\ (a + b + c)^{\frac{1}{2}}$$
$$\left\{\sqrt{(a + b)^2 - c^2} + \sqrt{(a + c)^2 - b^2} + \sqrt{(b + c)^2 - a^2}\right\}.$$
$$(x - a)^{\frac{3}{2}}(x - b)^{\frac{1}{2}}(x - c)^{\frac{1}{2}} + (x - b)^{\frac{3}{2}}(x - a)^{\frac{1}{2}}$$
$$(x - c)^{\frac{1}{2}} + (x - c)^{\frac{3}{2}}(x - a)^{\frac{1}{2}}(x - b)^{\frac{1}{2}}.$$

. Rationalize the denominator of the fraction—

$$\frac{1}{x - \sqrt{x^2 - 1}}.$$

Here the multiplier is $x + \sqrt{x^2 - 1}$, and the fraction becomes

$$\frac{x + \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)} = x + \sqrt{x^2 - 1}.$$

Similarly the following may be rationalized—

$$\frac{x}{\sqrt{x^2+1} - 1}; \quad \frac{\sqrt{a}}{\sqrt{a} + 1}; \quad \frac{3\sqrt{xy}}{x + \sqrt{y}}; \quad \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}};$$

$$\frac{1 + x - \sqrt{2x + x^2}}{1 + x + \sqrt{2x + x^2}}; \quad \frac{\sqrt{2+x} + \sqrt{x}}{\sqrt{2+x} - \sqrt{x}}; \quad \frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}};$$

$$\frac{a+b}{a + \sqrt{a^2 + ab}}; \quad \frac{\sqrt{1+a} - 1}{\sqrt{1-a} + 1}; \quad \frac{\sqrt{x^2 + y^2} + x}{\sqrt{x^2 + y^2} - x}; \quad \frac{1 + \sqrt{a^2 - 1}}{1 + 2x\sqrt{a^2 - 1}};$$

$$\frac{\sqrt{4x+1} + \sqrt{4x}}{\sqrt{4x+1} - 2\sqrt{x}}; \quad \frac{\sqrt{1 + ax^n} + \sqrt{1 - ax^n}}{\sqrt{1 + ax^n} - \sqrt{1 - ax^n}};$$

$$\frac{\sqrt{x} - \sqrt{x - (x^2 - ax)^{\frac{1}{2}}}}{\sqrt{x} + \sqrt{x - (x^2 - ax)^{\frac{1}{2}}}}; \quad \frac{1}{2\sqrt{2} - \sqrt{3}}; \quad \frac{3 + \sqrt{5}}{3 - \sqrt{5}}; \quad \frac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}};$$

$$\frac{3\sqrt{2} + 4\sqrt{7}}{5\sqrt{2} + 2\sqrt{7}}.$$

10. Find the numerical values of the following :—

$$\frac{1 + \sqrt{3}}{\sqrt{2} + \sqrt{3}}; \quad \frac{\sqrt{3} - \sqrt{2}}{3\sqrt{3} - 2\sqrt{2}}; \quad \frac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2 + \sqrt{3}}}; \quad \frac{2 + 4\sqrt{7}}{2\sqrt{7} - 1}$$

$$\frac{2 + \sqrt{3}}{3\sqrt{6} + 2\sqrt{12}}; \quad \frac{1}{\frac{1}{2}\sqrt{18} + 2}; \quad \frac{41 + 29\sqrt{2}}{7 + 5\sqrt{2}};$$

$$\frac{\sqrt{3} + 2}{\sqrt{2} + \sqrt{2 + 3^{\frac{1}{2}}}} + \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - \sqrt{2 - 3^{\frac{1}{2}}}}.$$

11. Rationalize the following :—

$$\frac{1}{1 + \sqrt{x} + \sqrt{y}}; \quad \frac{x - \sqrt{xy} + y}{x + \sqrt{xy} + y}; \quad \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{a + x} + \text{[illegible]}}$$

$$\frac{1 + \sqrt{2}}{1 + \sqrt{2} + \sqrt{3}}; \quad \frac{\sqrt{6}}{2\sqrt{2} - 3\sqrt{3} + 6\sqrt{6}}; \quad \frac{\sqrt[3]{28}}{\sqrt[3]{7} + \sqrt[3]{4}};$$

$$\frac{1}{2 - \sqrt[3]{3}}; \quad \frac{\sqrt[4]{3} - \sqrt[3]{5}}{\sqrt[4]{3} + \sqrt[3]{5}}; \quad \frac{\sqrt[3]{6}}{\sqrt[3]{4} - \sqrt[6]{2}}; \quad \frac{1 + \sqrt{3}}{1 - \sqrt[4]{3}}; \quad \frac{3^{\frac{1}{2}} 2^{\frac{1}{3}}}{2^{\frac{1}{3}} + 3^{\frac{1}{2}}};$$

$$\frac{x^{\frac{1}{4}}y^{\frac{1}{5}}}{2x^{\frac{3}{4}}-3y^{\frac{1}{5}}};\ \frac{(a-x)^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}-(a+x)^{\frac{1}{2}}};\ \frac{1-(1+x)^{\frac{1}{3}}}{1-(1-x)^{\frac{1}{3}}};\ \frac{\sqrt[3]{9}-5}{\sqrt[6]{9}+\sqrt{5}};$$

$$\frac{\sqrt[6]{9}-2\sqrt{2}}{\sqrt[6]{9}+2\sqrt{2}};\ \frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}.$$

In the following examples simplify each fraction separately before reducing them to a common denominator.

12. Simplify the following expressions :—

$$\frac{\sqrt{3}+2}{\sqrt{2}+\sqrt{2+3^{\frac{1}{2}}}}+\frac{\sqrt{3}-2}{\sqrt{2}-\sqrt{2-3^{\frac{1}{2}}}};$$

$$\frac{1}{a-\sqrt{a^2-1}}-\frac{1}{a+\sqrt{a^2-1}};\ \frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}-\frac{a-\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}};$$

$$\frac{(a^2+4)^{\frac{1}{2}}+(a^2-4)^{\frac{1}{2}}}{(a^2+4)^{\frac{1}{2}}-(a^2-4)^{\frac{1}{2}}}+\frac{(a^2-4)^{\frac{1}{2}}-(a^2+4)^{\frac{1}{2}}}{(a^2-4)^{\frac{1}{2}}+(a^2+4)^{\frac{1}{2}}};$$

$$\frac{x^{\frac{1}{3}}-y^{\frac{1}{3}}}{x-y}-\frac{1}{x^{\frac{2}{3}}+(xy)^{\frac{1}{3}}+y^{\frac{2}{3}}};\ \frac{1}{x+1+x^{-1}}+\frac{1}{1+x^{-1}+x^{-2}}$$

$$+\frac{1}{x^2+x+1}.$$

13. Extract the square roots of the following binomial surds :—

$2+\sqrt{3}$; $11-6\sqrt{2}$; $17-12\sqrt{2}$; $63+4\sqrt{143}$; $\frac{9}{4}+\sqrt{2}$;
$8-2\sqrt{7}$; $\frac{43}{36}-\frac{2}{3}\sqrt{3}$; $\frac{2}{99}(10-3\sqrt{11})$; $28+8\sqrt{10}$;
$37\sqrt{3}-60$; $11\sqrt{7}+2\sqrt{210}$; $\frac{23}{\sqrt{14}}+\frac{8}{\sqrt{2}}$; $\sqrt{11}-4\sqrt{\frac{7}{11}}$;
$2(a^2+b^2+ab+(a+b)\sqrt{a^2+b^2})$; $2(a+1)(a-\sqrt{a^2-1})$;
$2(a^2+b^2+c^2+(a+b+c)\sqrt{a^2+b^2+c^2})$.

14. Find the fourth root of $17-12\sqrt{2}$; $92+32\sqrt{7}$;
$\frac{1}{9}(73+28\sqrt{6})$; $33-20\sqrt{2}$; $14\sqrt{3}+24$.

15. Find the cube root of $584+195\sqrt{3}$; of $45-29\sqrt{2}$; $2+\sqrt{5}$;
$\frac{15}{4}\sqrt{3}-\frac{37}{8}$; the sixth root of $99-70\sqrt{2}$; $208-120\sqrt{3}$;
the eighth root of $577-408\sqrt{2}$; the ninth root of
$69826-40445\sqrt{3}$.

16. Show that $\sqrt{5\frac{1}{3}} = \sqrt{3} + \frac{1}{\sqrt{3}}$ and $\sqrt{4\frac{1}{6}} = \frac{\sqrt{3}}{2} + \frac{2}{\sqrt{3}}$; also, that $(\frac{1}{2})^{\frac{1}{2}}$ is less than $(\frac{2}{3})^{\frac{2}{3}}$; and $\sqrt[4]{14}$ greater than $\sqrt[3]{7}$.

# CHAPTER XII.

## QUADRATIC EQUATIONS OF ONE OR MORE UNKNOWN QUANTITIES.

**109.** *An equation is called Quadratic when the highest power of* x *contained in it is the second or square.*

Thus $3x^2 = 75$ (1.)
and $x^2-7x = 5x+13$ (2.)
are both quadratic equations.

**110.** *An equation in which only the second power of* x *appears is called a "pure" Quadratic.*

Thus (1.) is a pure quadratic.
It may be solved in the following manner :

$$3x^2 = 75$$

Divide both sides by 3,

$\therefore$ $x^2 = 25$

and $\sqrt{x^2} = \sqrt{25}$

Extracting the square root of both sides,

$$x = +5$$

also $x = -5$

for +5 and −5 are each a square root of 25.

It is usual to write the result of the square root thus :

$$x = \pm 5$$

Ex. (2.) $9-3x^2 = 4x^2-(3x^2+27)$

$\therefore$ $9-3x^2 = 4x^2-3x^2-27$

Transposing, &c. $-4x^2 = -27-9$

$\therefore$ $4x^2 = 36$

and $x^2 = 9$

$$x = \pm 3$$

**111.** To solve a pure Quadratic Equation, observe the following

RULE :—

(a.) *Clear the equation of fractions if necessary.*

(b.) *Collect all the terms involving* x$^2$ *on the first side of the equation, and the known quantities on the second side, and simplify.*

(c.) *Divide both sides by the co-efficient of* x$^2$.

(d.) *Extract the square root of both sides, and prefix the double sign.*

### Examples (LII.)

$7x^2-2=26$ ; $1-x^2=4x^2-24$; $2(1+x^2)=7(1-x^2)+4$

$8+3x^2-(11+2x^2)=0$; $2(x^2+1)+3(x^2+{}^2)+4(x2+3)-2=0$

$2x^2-\frac{2}{3}=8\frac{2}{3}-\frac{x^2}{3}$; $\frac{1}{4}(10x^2-1)=\frac{1}{3}(7x^2+6)+\frac{5}{12}$

$(x^2-1)-\frac{1}{2}(x^2-2)+\frac{1}{3}(x^2-3)-\frac{1}{4}(x^2-4)=1\frac{5}{7}$

$$\frac{(x+1)\,(x-3)}{2}=\frac{(x-1)\,(x-3)}{4}$$

$$\frac{x\,(x-1)\,(x-2)}{3}-\frac{(x-1)\,(x-2)\,(x-3)}{6}-\frac{2x\,(x-3)\,(x-4)}{12}=0$$

$$\frac{1}{4}\left(x^2-x+\frac{7}{12}\right)-\frac{3}{8}\left(x^2+x-\frac{5}{2}\right)=\frac{8-x}{8}+\frac{5}{24}$$

$$(x+a)\,(x-b)\,(x-c)+(x-a)\,(x+b)\,(x-c)+(x-a)\,(x-b)\,(x+c)$$
$$=(x+a)(x+b)(x-c)+(x+a)(x-b)(x+c)+(x-a)(x+b)(x+c)$$

**112.** *An equation in which both the first and second powers of* x *occur is called an adfected quadratic.*

Before giving examples of the solution of adfected quadratic equations, it will be necessary to discuss the process of "completing the square."

Now, we know that

$$(x+a)^2=x^2+2ax+a^2$$

that is, the expression

$$x^2+2ax+a^2$$

is a perfect square, and it is seen that the last term ($a^2$) is the square of half the coefficient of $x$, and that the coefficient of ($x^2$) is $+1$.

Now, suppose we take the expression

$$x^2+6x;$$

if it be required to make this a complete square, we must add the square of half the coefficient of $x$, viz. $(3)^2$;

it then becomes

$$x^2+6x+9,$$

which is the exact square of $x+3$.

Similarly to make $x^2+10a^2x$ a complete square, we must add $(5a^2)^2$ or $25a^4$.

Then

$$x^2+10a^2x+25a^4 = (x+5a^2)^2.$$

Again, to make $x^2-\frac{3}{2}px$, a complete square, $\frac{9}{16}p^2$ must be added, and the square is

$$x^2-\frac{3}{2}px+\frac{9}{16}p^2.$$

Hence then we may complete the square when the expression contains $x^2$ (the coefficient being $+1$) and $x$ with any coefficient by the following

### RULE.

*Add to the two given terms the square of half the coefficient of* x.

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## Examples LIII.

Complete the squares in the following expressions:

(1.) $x^2+4x$; $x^2-6x$; $x^2+x$; $x^2+3x$; $x^2+\frac{9}{4}x$; $x^2+\frac{x}{8}$

$x^2+ax$; $x^2-3bx$; $x^2-8abx$; $x^2-3px$; $x^2-2a^2x$; $x^2+9p^3x$.

(2.) Complete the square in the expression

$$x^6-4a^3x^3$$

*Working.* The quantity to be added is

$$(2a^3)^2 \text{ or } 4a^6,$$

and the complete square is

$$x^6-4a^3x^3+4a^6$$

*Note.* Whenever the index of the letter in one of the terms is double of its index in the other term of a binomial, the square may be completed by the above Rule, as in the instance just given.

(3.) Make the following expressions complete squares:

$x^4+x^2$; $x^6-ax^3$; $x^8-a^2x^4$; $x^4-a^2x^2$; $x^{10}-b^5x^5$; $x^6+10p^2q^2x^3$

$x^4-\frac{1}{2}a^2x^2$; $x^{2p}-3a^px^p$; $x^{2m}-\frac{p^mx^m}{3}$; $x^6+\frac{abcdx^2}{m^2}$; $x^4-\frac{3mx^2}{2n}$

$x^{10}-\frac{2amx^5}{5bn}$.

(4.) Write down the "square root" of each of the completed squares in the above examples.

Let it be required to solve the equation

$$3x^2-12x+4 = -5$$

*Working.*

By transposition

$$3x^2-12x = -9$$

Dividing both sides by 3

$$x^2-4x = -3$$

Completing the square by adding $(-2)^2$ or 4 to each side,

$$x^2-4x+4 = 4-3$$

or $$x^2-4x+4 = 1$$

Extracting the square root of both sides,

$$x-2 = \pm 1$$

$\therefore$ $$x = 2\pm 1$$

$$\left.\begin{aligned}&\text{Taking the + sign} && x = 3\\ &\text{Taking the − sign} && x = 1\end{aligned}\right\}$$

Hence 3 and 1 are the two roots of $x$ in the above equation.

Again to solve the equation

$$ax^2+bx-c = 0$$

Proceeding by the same steps as above, we have

By transposition $$ax^2+bx = c$$

Dividing by $a$ $$x^2+\frac{bx}{a} = \frac{c}{a}$$

Completing the square $$x^2+\frac{bx}{a}+\frac{b^2}{4a^2} = \frac{c}{a}+\frac{b^2}{4a^2}$$

or $$x^2+\frac{bx}{a}+\frac{b^2}{4a^2} = \frac{4ac+b2}{4a^2}$$

Extracting the square root,

$$x+\frac{b}{2a} = \pm\frac{\sqrt{4ac+b^2}}{2a}$$

$$\left.\begin{aligned}&\therefore && x = -\frac{b}{2a}+\frac{\sqrt{4ac+b^2}}{2a}\\ &\text{and} && x = -\frac{b}{2a}-\frac{\sqrt{4ac+b^2}}{2a}\end{aligned}\right\}$$

are the two values of $x$.

These values may be put in the more convenient form

$$\left.\begin{aligned}x &= -\frac{b-\sqrt{4ac+b^2}}{2a}\\ x &= -\frac{b+\sqrt{4ac+b^2}}{2a}\end{aligned}\right\}$$

Solve by a similar method the equation

$$x^2+px+q = 0.$$

The two values will be found to be

$$x = -\frac{p}{2} \pm \frac{\sqrt{p^2-4q}}{2}$$

## 113. For the general solution of a Quadratic Equation, the following Rule may be given.

(a.) *Apply the rule of transposition to bring all the terms containing* $x^2$ *and* $x$ *to the first side of the equation, and all the other terms to the second side.*

(b.) *Simplify both sides by collecting coefficients.*

(c.) *Divide both sides by the coefficient of* $x^2$.

(d.) *Complete the square by adding to both sides the square of half the coefficient of* $x^2$.

(e.) *Extract the square root of both sides, placing the double sign* ± *on the second side.*

(f.) *Write down the two resulting values of* $x$.

### Examples LIV.

(1.) Solve the following equations:

$3x^2+10 = 24x-11$; $2x^2+x = 3$; $3-7x = 4x^2+6$

$x^2-11x-12 = 0$.; $(2x+3)^2 = 7\,(2-x)$; $(5-x)^2+8\,(7-x) = 0$

$x^2+\frac{x}{2}+\frac{1}{4} = \frac{3}{16}$; $x^2-\frac{1}{6}\,(x-3) = \frac{7}{12}\,(x-1)+22\frac{1}{3}$

$(x-1)^2+(x-2)^2+(x-3)^2 = 5$; $(2x-6)^2-9\,(x+4)+63 = 0$

$(2x-5)^2-81\,(x+4)^2 = 0$; $x^2 = 49\,(x-3)^2$; $(3x-1)\,(4x+1) = 0$

$(7x-14)\,(6x+13) = 0$.

*Note.* The two last equations may be solved by a different method from the usual one.

For the equation $(3x-1)\,(4x+1) = 0$ states that the product of the two factors $(3x-1)$ and $(4x+1)$ must equal 0; and it is required to find what values must be substituted for $x$ to produce this result.

Now it is evident that if *one* only of the factors be made equal to 0, then the product of the two must equal 0, whatever the other factor be: hence, if we can find those values of $x$ which make each factor separately equal to 0, both of those values will be roots of the equation.

Therefore, to solve the equation above,

(1.) put $3x-1 = 0$

$\therefore$ $3x = 1$

and $x = \frac{1}{3}$

(2.) put $4x+1 = 0$

$\therefore$ $4x = -1$

and $x = -\frac{1}{4}$

that is, $\frac{1}{3}$ and $-\frac{1}{4}$ are the two roots of the above equation.

Generally if an equation can be put in the form

$$(x-a)\ (x-b)\ (x-c) \ldots\ldots = 0$$

all the roots of the equation can be at once determined by putting each factor separately $= 0$,

that is, $x-a = 0$, which gives $x = a$

$x-b = 0$ ,, $x = b$

$x-c = 0$ ,, $x = c$

&c., &c., &c.

(2.) $(x+17)\ (x+3) = (x+11)\ (2x+4)$

$$(2x-3)(3x-4) = \frac{(3-x)\ (8-3x)}{2}; \quad \left(x-\frac{1}{2}\right)(2+x) = 9$$

$$\frac{1}{2}(3-x)\left(x-\frac{3}{4}\right)+\frac{11}{16} = 0; \quad \frac{x^2+1}{10}-\frac{x-2}{9}+\frac{3x+1}{4} = 1$$

$$\frac{x^2-1}{2}+\frac{x^2+2}{4} = 4\,(x-1); \quad \frac{x^2}{7}+2\frac{17}{28} = -\frac{7}{25}x$$

$$\frac{(x-4)\ (x-12)}{3} = \frac{(2x+11)\ (x-10)}{37}$$

$$(x+a)(x+b)(x-c)+(x+a)(x-c)(x+d)+(x+b)(x-c)(x+d) = 0$$

Solve the equation

$$\frac{9}{x+4} = \frac{4}{x-1} - \frac{5}{24}$$

*Working.* To solve this, we must first find the L. C. M. of the denominators, and multiply both sides of the equation by it.

L. C. M. $= 24\ (x+4)\ (x-1)$

Multiplying both sides of the equation by this L. C. M., it becomes

$$\frac{9\times 24\ (x+4)\ (x-1)}{x+4} = \frac{4\times 24\ (x+4)\ (x-1)}{x-1} - \frac{5\ (x+4)\ (x-1)\ \times 24}{24}$$

or $24\times 9\ (x-1) = 4\times 24\ (x+4) - 5\ (x+4)\ (x-1)$

or $216\ (x-1) = 96\ (x+4) - 5\ (x+4)\ (x-1)$,

which may be solved in the usual way.

Again, take the equation

$$\frac{x}{4}-\frac{x}{6}=\frac{3}{2x}-\frac{1}{x}$$

Here the L. C. M. = $12x$, and multiplying every term by $12x$, we get

$$\frac{12x\,.\,x}{4}-\frac{12x\,.\,x}{6}=\frac{12\times 3x}{2x}-\frac{12x}{x}$$

$$\text{or} \qquad 3x^2-2x^2 = 18-12$$

$$\therefore \qquad x^2 = 6$$

$$\text{and} \qquad x = \pm\sqrt{6}$$

The following examples must be solved by a similar method:

(3.)

$$x+1-\frac{1}{x+1} = 2x-\frac{4}{3}$$

$$x+2+\frac{1}{x+2} = x-\frac{1}{x}$$

$$\frac{3}{x^2-1}-\frac{5}{x^2+1} = 0$$

$$1+\frac{3}{x-4} = 2-\frac{3}{x+4}$$

$$\frac{100}{x^2+1} = \frac{80}{x^2-1}$$

$$\frac{x}{5}-\frac{5}{x} = \frac{x}{1-5x}-\frac{1-5x}{x}$$

$$\frac{80}{3x-1}-\frac{x+3}{10}+\frac{x-17}{5} = 0$$

$$\frac{x+1}{4}-\frac{10}{x-9} = \frac{x-10}{3}$$

$$\frac{1}{x+1}+\frac{5}{x-1}+\frac{20}{x^2-1} = 0$$

$$\frac{3}{x-1}-\frac{4}{x-2}+\frac{9}{x^2-3x+2} = 0$$

$$\frac{3}{x^2-1}-\frac{2}{x^2+1} = \frac{21}{4\,(x^4-1)}$$

$$\frac{7}{x+2}-\frac{3}{x-1} = \frac{3}{x^2+x-2}$$

$$\frac{x^2-7}{x+3} = x+\frac{1}{x}-3$$

$$\frac{x}{x+1}-\frac{x+1}{x} = \frac{2\,(x-2)}{x^2-10}-\frac{1}{2\,(x^2-10)}$$

$$2\left(1-\frac{2}{x+1}\right)+3\left(2+\frac{8}{x-4}\right) = \frac{3}{2x+3}-1$$

$$\frac{4\,(x-5)}{x}-3=\frac{4x}{x-5}+12$$

$$\frac{5+x}{4}-\frac{4-x}{4x-5}=2x-\frac{3x+20}{7}$$

$$\frac{1}{2\,(1-x)}-\frac{4}{3}=-\frac{1}{2\,(1+x)}$$

$$\frac{14}{7-x}-\frac{2x}{7+x}=\frac{29}{10}$$

$$\frac{3x-5}{3x+5}+1=\frac{3x+5}{3x-5}+\frac{41}{176}$$

$$\frac{2x}{x-7}-\frac{2x}{x+7}=\frac{7x}{x^2-73}$$

$$\frac{3\,(x+2)}{4x+3}+7x-11=\frac{14x^2-27}{2x+3}$$

$$\frac{x-9}{x-12}-1\frac{5}{12}=\frac{3}{x-6}$$

$$\frac{4}{x+2}-\frac{12}{x+6}+\frac{5}{x+4}=0$$

$$\frac{7}{x+2}-\frac{3}{2x}=\frac{5}{2\,(x+1)}$$

$$\frac{1}{x+1}-\frac{1}{x-1}=\frac{1-4x}{2x-1}$$

$$\frac{1}{x}-\frac{1}{x+1}=\frac{3}{10\,(x+2)}$$

$$\frac{3}{5-2x}+\frac{1}{2-x}=\frac{4}{x+1}$$

$$\frac{x+1}{x-1}-\frac{x-2}{x+2}=\frac{9}{5}$$

$$\frac{2x+1}{2x-1}+\frac{3x-2}{3x+2}=\frac{13}{6}$$

$$\frac{3-4x}{3\,(3-x)}+\frac{1}{2\,(1-x)}=1\frac{1}{3}$$

$$\frac{3}{x-7}+\frac{7}{x+6}=\frac{x^2-28}{x^2-x-42}$$

$$\frac{x+3}{x^2-3x+2}+\frac{x+1}{x^2-4x+3}=\frac{5x-3}{3\,(x^2-5x+6)}$$

$$\frac{3x}{x^2-1}+\frac{6x}{x^2-3x+2}+\frac{x-3}{x^2-x-2}=\frac{11x^2-2}{x^3-2x^2-x+2}$$

$$x^2-2ax=2a+1$$

$$x^2+(a+b)x-\frac{ab}{2}-\frac{3b^2}{4}=0$$

$$adx-acx^2 = bcx-bd$$

$$b^2x^2-2bx = \frac{b^2}{a^2}-1$$

$$\frac{a-x}{a} = \frac{3a-2x}{8x}$$

$$abx^2-a^2cx-b^2cx+abc^2 = 0$$

$$\frac{1}{ab-ax}+\frac{1}{bc-bx}-\frac{1}{cd-ax} = 0$$

$$x^2(b^2-c^2)-2ac^2x+2ab^2x-a^2c^2 = 0$$

## 114. Other Equations which may be solved as Quadratics.

Take the equation

$$x4+4x^2 = 320;$$

To solve this we may proceed thus:—

Add to both sides

$$\left(\frac{4}{2}\right)^2 \text{ or } 4,$$

Then $x^4+4x^2+4 = 324.$

The first side is now a complete square;

Taking the square root of both sides, we get

$$x^2+2 = \pm 18$$

$$x^2 = 16 \text{ or } -20$$

Whence $x = \pm 4$

and $x = \pm\sqrt{-20}$

Here the first part of the solution is found by completing the square as in an ordinary quadratic.

Again, let $3x^{2m} - 7x^m = 10$ be the equation;

Divide both sides by 3;

$$\therefore \quad x^{2m}-\frac{7}{3}x^m = \frac{10}{3}.$$

Add $\left(\frac{7}{6}\right)^2$ to both sides.

Then $x^{2m}-\frac{7}{3}x^m+\frac{49}{36} = \frac{169}{36}.$

Extract the square root of both sides.

$$x^m - \frac{7}{6} = \pm\frac{13}{6}.$$

$$\therefore \qquad x^m = \frac{10}{3} \text{ or } 1,$$

And extracting the $m^{\text{th}}$ of both sides. we have

$$x = \sqrt[m]{\frac{10}{3}} \text{ or } 1.$$

**115.** From the last example it may be seen that the method of solving Quadratics may be applied to all equations which contain only *two* powers of $x$, such that the index of the greater power is double that of the less.

Also, note that the quadratic method cannot generally be applied to equations which contain more than two different powers of $x$; nor to those which contain two powers of $x$, such that the index of the greater is not exactly double that of the less.

Let the pupil satisfy himself of this by trying to complete the square in the following equations:

$$x^4 + 2x = 10x^2 + 5$$
$$4x^6 - 8x^2 + 4 = 0.$$

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### Examples LV.

$x^4 - 6x^2 - 27 = 0$; $x^6 + 14x^3 = 1107$;

$x^6 - 38x^3 - 864 = 0$; $\dfrac{x^2}{x^2-4} + 6 = 351\,\dfrac{(x^2-4)}{x^2}$;

$\dfrac{14x^2+8}{21} - \dfrac{x^2+4}{8x^2-11} = \dfrac{2x^2}{3}$;

$\dfrac{2x^2+1}{2x^2-1} - 7\dfrac{23}{77} = -\dfrac{9(2x^2-1)}{2x^2+1}$;

$x^2(1-x^2) = \dfrac{4}{25}$; $(x^2-5)^2 - (x-3)^2 = (x+1)^2$;

$\left(\dfrac{x}{x+1}\right)^2 + \left(\dfrac{x}{x-1}\right)^2 = 12$; $x^4 - 3x^3 + 6x = 4$;

$\left(\dfrac{x}{x+1}\right)^2 + \left(\dfrac{x}{x-1}\right)^2 = a(a-1)$;

$\dfrac{x^{p+1}}{x^{p-1}} - \dfrac{x^{p-1}}{x^{p+1}} = 2$; $x^{\frac{p}{p+q}} - \dfrac{8}{x^{\frac{2p}{p+q}}} = \dfrac{7}{x^{\frac{p}{p+q}} - 2}$.

**116.** It has been remarked above that equations cannot *erally* be solved as Quadratics when they contain more n two different powers of the unknown quantity. There , however, some cases in which they can be so solved, in the following examples.

Let $(x^2 - 7x)^2 - 10x^2 + 70x + 16 = 0$ be the equation.

This may be put in the form

$$(x^2 - 7x)^2 - 10(x^2 - 7x) = -16$$

Add 25 to both sides,

$$\therefore \quad (x^2 - 7x)^2 - 10(x^2 - 7x) + 25 = 9.$$

Extract square root,

$$(x^2 - 7x) - 5 = \pm 3$$

$$\therefore \quad x^2 - 7x = 8 \text{ or } 2.$$

We must now solve in succession the two equations

$$x^2 - 7x = 8$$

And $$x^2 - 7x = 2.$$

The first gives $x = 8$ or $-1$,

The second gives $x = \dfrac{7 \pm \sqrt{57}}{2}$,

So that there are four roots to the equation.

Next let $x^2 + \sqrt{x^2 + 11} = 31$,

Add 11 to both sides,

$$\therefore \quad (x^2 + 11) + \sqrt{x^2 + 11} = 42,$$

Add $\frac{1}{4}$ to complete the square,

$$\therefore \quad x^2 + 11) + \sqrt{x^2 + 11} + \tfrac{1}{4} = \frac{169}{4}.$$

Extract square root $\sqrt{x^2 + 11} + \frac{1}{2} = \pm \dfrac{13}{2}$,

And $\sqrt{x^2 + 11} = -\frac{1}{2} \pm \dfrac{13}{2}$.

Or, $\sqrt{x^2 + 11} = -6$ or $-7$.

Square both sides $x^2 + 11 = 36$ or $49$

$\therefore$ $x^2 = 25$ or $38$

And $x = \pm 5$ or $\pm\sqrt{38}$.

The solution gives four values of $x$; it will be found, owever, by trial that $\pm 5$ are the only two values that satisfy the equation. The two values $\pm\sqrt{38}$ belong to the equation

$$x^2 - \sqrt{x^2+11} = 31.$$

It is therefore necessary in equations that contain surd expressions to examine, by actual substitution, whether the roots obtained will satisfy the equation, and to retain only those which do satisfy it.

Let $x^2 + \frac{9}{x^2} - 6x + \frac{18}{x} + 14 = 0$ be the equation.

This may be put in the form

$$x^2 + 6 + \frac{9}{x^2} + 6\left(x + \frac{3}{x}\right) + 8 = 0$$

Or, $\left(x + \frac{3}{x}\right)^2 - 6\left(x + \frac{3}{x}\right) = -8$

$\therefore$ $\left(x + \frac{3}{x}\right)^2 - 6\left(x + \frac{3}{x}\right) + 9 = 1$

And $\left(x + \frac{3}{x}\right) - 3 = \pm 1,$

$$\left.\begin{aligned} \therefore\quad x + \frac{3}{x} &= 4 \\ \text{And}\quad x + \frac{3}{x} &= 2 \end{aligned}\right\}$$

which are ordinary Quadratics, from which the four values of $x$ may be found.

Take the equation $x^4 - 2x^3 + x - 1 = 0.$

This may be put in the form

$$x^2(x^2 - 2x + 1) - x(x-1) = 1$$

Or, $\{x(x-1)\}^2 - x(x-1) = 1$

$\therefore$ $\{x(x-1)\}^2 - x(x-1) + \frac{1}{4} = \frac{5}{4}$

And $x(x-1) - \frac{1}{2} = \pm\frac{\sqrt{5}}{2},$

$$\left.\begin{aligned} \text{Or,}\quad x^2 - x &= \frac{1+\sqrt{5}}{2} \\ \text{And}\quad x^2 - x &= \frac{1-\sqrt{5}}{2} \end{aligned}\right\}$$

are the two equations to be solved.

Taking the first,

$$x^2 - x = \frac{1 + \sqrt{5}}{2}$$

$$\therefore \quad x^2 - x + \tfrac{1}{4} = \frac{3 + 2\sqrt{5}}{4}$$

$$x - \tfrac{1}{2} = \pm \frac{\sqrt{3 + 2\sqrt{5}}}{2}$$

And $$x = \frac{1 \pm \sqrt{3 + 2\sqrt{5}}}{3}.$$

Similarly, the other may be solved.

Lastly, let $25x\,(x - \sqrt{1 - x^2}) = 4,$

Or, $25x^2 - 25x\sqrt{1 - x^2} = 4.$

Transposing $25x\sqrt{1 - x^2} = 25x^2 - 4.$

Squaring both sides, we get

$$625x^2(1 - x^2) = 625x^4 - 200x^2 + 16,$$

Or, $- 1250x^4 + 825x^2 = 16,$

$$\therefore \quad x^4 - \frac{33}{50}x^2 = -\frac{8}{625},$$

which may be solved as in former cases.

The four values of $x$ will be found to be

$$\pm\frac{4}{5} \text{ and } \pm\frac{\sqrt{2}}{10}.$$

And of these four, only $+\frac{4}{5}$ and $-\frac{\sqrt{2}}{10}$ satisfy the equation $25x^2 - 25x\sqrt{1 - x^2} = 4.$

The two roots $-\frac{4}{5}$ and $+\frac{\sqrt{2}}{10}$ belong to the equation $25x^2 + 25x\sqrt{1 - x^2} = 4.$

It will in fact be found that the equation

$$x^4 - \frac{33}{50}x^2 = -\frac{8}{625},$$

which we have solved, may be obtained from squaring either of the two equations

$$25x^2 - 25x\sqrt{1 - x^2} = 4$$

$$25x^2 + 25x\sqrt{1 - x^2} = 4,$$

(Let the pupil prove this by squaring)

and the four roots belong to these two equations, two to one and two to the other.

These remarks apply to the equation solved in p. (148).

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### Examples LVI.

(1.) $$x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4;$$

$$2x^2 + \sqrt{2x^2 + 1} = 10;$$

$$x - 1 = \sqrt{x + 5};$$

$$42 - x - \frac{8}{x} = \left(x + \frac{8}{x}\right)^2;$$

$$x^2 - 2x - 5\sqrt{2x^2 - 5x + 6} = \frac{x + 33}{2};$$

$$x^{12} + 4x^6 = 5;$$

$$x^6 + 3 = \frac{4}{x^6};$$

$$8x^2 - 13 - \sqrt{6x^3 + 52x^2} = \frac{3x}{2};$$

$$\sqrt{5 + x} + \sqrt{3 - x} = 4;$$

$$2\sqrt{x} = \sqrt{10x + 12} - \sqrt{3x + 2};$$

$$\sqrt[3]{x^3 + 1} + \sqrt[3]{x^3 - 1} = x;$$

$$2(x^4 + 1) - (x + 1)^4 = 0;$$

$$\sqrt{\frac{x + 4}{x - 4}} + x + 4 = \frac{12}{x - 4};$$

$$\sqrt{4x - \sqrt{7x + 2}} = 1 - 2\sqrt{x};$$

$$1 + \sqrt{\frac{a - x}{a + x}} = \frac{12x}{5(a + x)};$$

$$x^4 - 2x^3 + 2x^2 + x = 6;$$

$$\frac{x - \sqrt{x^2 - 16}}{x + \sqrt{x^2 - 16}} = \frac{4}{x};$$

$$x^2(x^2 - 18) = 4(x - 12);$$

$$x^2 - 4\sqrt{x} = 3(x + 1);$$

$$x^2(4x^2 - 1) = 7x + 12;$$

$$2x^2 - x - 2x\sqrt{1 - x^2} = \tfrac{3}{2};$$

Let $$8x^3 + 64x = 72;$$

$$\therefore \quad x^3 + 8x = 9.$$

This may be put in the form

$$x^3 - x + 9x - 9 = 0;$$

Or, $$x(x^2 - 1) + 9(x - 1) = 0$$

$$\therefore \quad x(x - 1)(x + 1) + 9(x - 1) = 0$$

And $$(x - 1)\{x^2 + x + 9\} = 0.$$

Hence we have the two equations

$$x - 1 = 0 \text{ giving } x = 1,$$

And $$x^2 + x + 9 = 0,$$

which is an ordinary quadratic, and its two roots are

$$\tfrac{1}{2}(-1 \pm \sqrt{-35}).$$

When an equation containing the third power of $x$ as the highest is proposed for solution, it is necessary to collect the whole expression on the first side of the equation, to put this $= 0$; then to break the expression into two factors, one a simple factor, the other a quadratic factor, and to put each of these factors separately $= 0$. Then three roots will result.

(2.)
$$x^2 - 4 = (x + 2)(x^2 + 8);$$

$$x^2(x + 4) + x + 4 = 2 - 2x(x + 4);$$

$$(x - 4)^2 + 1 = \frac{2}{x} - 2(x - 4);$$

$$2x^3 = x^2 + 1;$$

$$x^3 - 2x^2 + x = 4(1 - x^2);$$

$$x + \frac{1}{1 - x} - \frac{1}{1 - x} - c - \frac{1}{1 - c} + \frac{1 - c}{c} = 0;$$

$$x^3 - 3x = 2;$$

$$2x^3 - 3x^2 + 5 = 0;$$

$$x^{\frac{3}{2}} + x(1 - \sqrt{a}) + 6\sqrt{a} = x^{\frac{1}{2}}\sqrt{a^{\frac{1}{2}} + 6};$$

$$2x^2 + 17x - 1 = (8x - 2)\sqrt{9 + x}.$$

**117.** The following solutions are given as examples of the method in which quadratic equations may be solved when they contain surds.

(1.)
$$2x^{\frac{1}{2}} + (4x + \sqrt{7x + 2})^{\frac{1}{2}} = 1$$

$$\therefore \quad (4x + \sqrt{7x + 2})^{\frac{1}{2}} = 1 - 2x^{\frac{1}{2}}.$$

Squaring $4x + \sqrt{7x+2} = 1 - 4x^{\frac{1}{2}} + 4x$

Or, $7x + 2 = (1 - 4x^{\frac{1}{2}})^2$

$7x + 2 = 1 - 8x^{\frac{1}{2}} + 16x$

$\therefore$ $9x - 8x^{\frac{1}{2}} = +1$

Or, $x - \frac{8}{9}x^{\frac{1}{2}} = \frac{1}{9}.$

Comp. Sq. $x - \frac{8}{9}x^{\frac{1}{2}} + \frac{16}{81} = \frac{25}{81}.$

Ex. sq. root $x^{\frac{1}{2}} - \frac{4}{9} = \pm\frac{5}{9}$

$\therefore$ $x^{\frac{1}{2}} = 1 \text{ or } \frac{1}{9}.$

Squaring again $x = 1 \text{ or } \frac{1}{81}.$

In this example, the most complicated surd expression is taken by itself on one side of the equation, and the simpler quantities are transposed to the other side; and, in general, this is the best method for equations of this form.

(2.) $3x^{\frac{3}{2}} - x^{-\frac{3}{2}} + 2 = 0.$

Multiply all through by $x^{\frac{3}{2}}$, we get

$3x^3 - 1 + 2x^{\frac{3}{2}} = 0,$

Or, $x^3 + \frac{2}{3}x^{\frac{3}{2}} = \frac{1}{3}$

$\therefore$ $x^3 + \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{9} = \frac{4}{9}$

And $x^{\frac{3}{2}} + \frac{1}{3} = \pm\frac{2}{3}$

$\therefore$ $x^{\frac{3}{2}} = \frac{1}{3} \text{ or } -1,$

Squaring $x^3 = \frac{1}{9} \text{ or } 1,$

Taking cube root $x = \sqrt[3]{\frac{1}{9}} \text{ or } 1.$

From this example it appears that negative indices should first be got rid of, unless both the indices are negative, and one is double of the other.

(3.) $$(a+x)^{\frac{1}{4}} + (a-x)^{\frac{1}{4}} = b^{\frac{1}{4}}.$$

Squaring both sides, and simplifying, we get

$$(a+x)^{\frac{1}{2}} + 2\,(a^2-x^2)^{\frac{1}{4}} + (a-x)^{\frac{1}{2}} = b^{\frac{1}{2}},$$

Or, $$(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}} = b^{\frac{1}{2}} - 2\,(a^2-x^2)^{\frac{1}{4}}.$$

Squaring again,

$$a + x + 2\,(a^2-x^2)^{\frac{1}{2}} + a - x = b - 4b^{\frac{1}{2}}\,(a^2-x^2)^{\frac{1}{4}} + 4\,(a^2-x^2)^{\frac{1}{2}},$$

Or, $$-\,2\,(a^2-x^2)^{\frac{1}{2}} + 4\,b^{\frac{1}{2}}\,(a^2-x^2)^{\frac{1}{4}} = b - 2a.$$

Here we have a quadratic in $(a^2-x^2)^{\frac{1}{4}}$; when solved, it gives

$$(a^2-x^2)^{\frac{1}{4}} = b^{\frac{1}{2}} \pm \left(\frac{2a+b}{2}\right)^{\frac{1}{2}}.$$

Raise both sides to the fourth power, we find a value of $x^2$; take the square root, and the value of $x$ becomes known.

(4.) $$(x+a)^{\frac{1}{2}} - (x-a)^{\frac{1}{2}} = c^{\frac{1}{2}}. \qquad (1)$$

Now $$(x+a) - (x-a) = 2a.$$

Dividing this latter identity by the equation (1), we get

$$(x+a)^{\frac{1}{2}} + (x-a)^{\frac{1}{2}} = \frac{2a}{c^{\frac{1}{2}}}. \qquad (2)$$

Adding (1) and (2) together, we find

$$2\,(x+a)^{\frac{1}{2}} = c^{\frac{1}{2}} + \frac{2a}{c^{\frac{1}{2}}}$$

$$= \frac{2a+c}{c^{\frac{1}{2}}}.$$

Squaring both sides $$4\,(x+a) = \frac{(2a+c)^2}{c},$$

from which $x$ is known.

This method may always be applied when the sum of two square roots, or their difference, is given, and it is always the best method of solution, if the coefficient of $x$ be the same in each of the radicals.

(5.) $$\frac{a + x + \sqrt{2ax + x^2}}{a + x - \sqrt{2ax + x^2}} = \frac{b+c}{b-c}.$$

To simplify this, apply Prop. (viii). Proportion, we get

$$\frac{a+x}{\sqrt{2ax+x^2}} = \frac{b}{c},$$

Clear of fractions, and square both sides, and the equation will be an ordinary quadratic, and we shall find

$$x = -a\left(1 \mp \frac{b}{\sqrt{b^2 - c^2}}\right).$$

(6.) $$\frac{c - x}{c^{\frac{1}{2}} + (c - x)^{\frac{1}{2}}} + \frac{c + x}{c^{\frac{1}{2}} - (c + x)^{\frac{1}{2}}} = c^{\frac{1}{2}}.$$

Here we shall simplify each fraction separately; multiply the numerator and denominator of the first fraction by $c^{\frac{1}{2}} - (c - x)^{\frac{1}{2}}$, and those of the second by $c^{\frac{1}{2}} + (c + x)^{\frac{1}{2}}$, we get

$$\frac{(c - x)c^{\frac{1}{2}} - (c - x)^{\frac{3}{2}}}{x} - \frac{(c + x)c^{\frac{1}{2}} + (c + x)^{\frac{3}{2}}}{x} = c^{\frac{1}{2}}.$$

Clearing of fractions, and simplifying, we get

$$(c + x)^{\frac{3}{2}} + (c - x)^{\frac{3}{2}} = -3c^{\frac{1}{2}}x.$$

Squaring and simplifying

$$2(c^2 - x^2)^{\frac{3}{2}} = 3cx^2 - 2c^3.$$

Square again and multiply.

Then we find $\quad x = 0$, or $x = \pm \dfrac{c\sqrt{3}}{2}$.

(7.) $$2a.x^{\frac{m+n}{2mn}} = x^{\frac{1}{m}} + x^{\frac{1}{n}}.$$

Divide both sides by $x^{\frac{1}{m}}$,

Then $$2a.x^{\frac{m-n}{2mn}} = 1 + x^{\frac{m-n}{mn}},$$

Or, $$x^{\frac{m-n}{2mn}} - 2a.x^{\frac{m-n}{2mn}} = -1,$$

which may be solved as a quadratic, since one index is double of the other. The result is

$$x = \left\{a \pm (a^2 - 1)^{\frac{1}{2}}\right\}^{\frac{2mn}{m-n}}.$$

The methods adopted in Examples (5) and (6) for clearing of fractions are often useful, and should be carefully studied.

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## Examples LVII.

(1.) $2\sqrt{49 - 8x} = 8 - \sqrt{5x + 6}$; $\sqrt{ax^2 + bx - c} + ax - a = 0$;
$\sqrt{3x + 10} = 5 - \sqrt{2x - 3}$; $x + \sqrt{x + 2} = \frac{7}{4}$;

$$\sqrt{x+4p} = 2\sqrt{x+q} - \sqrt{x}; \quad x+1 = \sqrt{2(x+1)} + 4;$$

$$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 198; \quad 3\sqrt{112-8x} = 19+\sqrt{3x+7};$$

$$\frac{1}{x} + \frac{1}{x^{\frac{1}{2}}} - 6 = 0; \quad \sqrt{(x+3c)^2+a^2} + \sqrt{(x-3c)^2+a^2} = 3c;$$

$$\frac{x+\sqrt{x^2-16}}{x-\sqrt{x^2-16}} = (x-3)^2; \quad \sqrt{\frac{b}{a+x}} + \sqrt{\frac{c}{a-x}} = \sqrt[4]{\frac{4bc}{a^2-x^2}};$$

$$ax - 3b(x-1) + 2\sqrt{b^2x + abx^2} = 0;$$

$$2x^2 + 5(x+1) + \sqrt{x^2+3x-1} + x = 0;$$

$$(x+\sqrt{x^2-1})^{\frac{1}{2}} - (x-\sqrt{x^2-1})^{\frac{1}{2}} = 2\sqrt{x^2-1};$$

$$(\sqrt{x}+\sqrt{c-x})^2 = a(\sqrt{x}-\sqrt{c-x}); \quad 2x^{\frac{1}{4}} + 2x^{-\frac{1}{4}} = 5;$$

$$\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}-\sqrt{3}} = \sqrt{\frac{108}{x}}; \quad \frac{x^{\frac{1}{3}}+4}{x^{\frac{1}{3}}+6} = \frac{x^{\frac{2}{3}}+20}{x^{\frac{2}{3}}+28};$$

$$2\sqrt{x^2+x} + 2x = 5 - \sqrt{x} - \sqrt{1+x};$$

$$\frac{1}{1+x+x^2} + \frac{x}{1+x+x^{-1}} + \frac{x^2}{1+x^{-1}+x^{-2}} = 0;$$

$$\left(\frac{abx+c^2}{bc} + \sqrt{\frac{4ax}{b}}\right)^{\frac{1}{2}} - \left(\frac{abx+c^2}{bc} - \sqrt{\frac{4ax}{b}}\right)^{\frac{1}{2}} = \frac{ax}{c} + 1;$$

$$\left(\frac{x}{2}+6\right)^{\frac{1}{2}} - \left(\frac{x}{2}-6\right)^{\frac{1}{2}} = 2\left(\frac{x}{3}\right)^{\frac{1}{2}}.$$

---

Some of the following will reduce to simple equations, and have therefore but one root.

(2.) $\sqrt{x+1} + \sqrt{x-12} = 13; \quad 1 + \frac{1}{\sqrt{x+1}} = \sqrt{x-1};$

$$\frac{x}{\sqrt{b+x}+1}(\sqrt{1-x}+1) = \frac{12}{25};$$

$$21 - 5\sqrt{x} = \frac{10\sqrt{x}}{105\sqrt{x}-2};$$

$$\sqrt{\frac{(x+c)^2-a^2}{2cx}} = \frac{\sqrt{3}+1}{2};$$

$$x^{\frac{3}{5}} - (x^{\frac{1}{5}} - 5)^3 = 35;$$

$$x^2 - 4x + 5 = \sqrt{11v^2 - 52x + 61};$$

$$3x + \sqrt{2x - 2} = 7;$$

$$x(1 - x) + 3\sqrt{2x^2 - 6x + 5} = 25 + 2x;$$

$$\frac{(x^2 - x\sqrt{2} + 4)^2}{8} = \frac{x}{2}(x^2 + x\sqrt{2} + 4) - 1 - 2\sqrt{2};$$

$$\left(x + \frac{1}{x}\right)^{\frac{1}{q}} - 5\left(x + \frac{1}{x}\right)^{\frac{1}{2q}}\left(x - \frac{1}{x}\right)^{\frac{1}{2q}} + 6\left(x - \frac{1}{x}\right)^{\frac{1}{q}} = 0;$$

$$\sqrt[3]{x + 1} - \tfrac{1}{2}\sqrt[3]{8x - 2q} = \tfrac{1}{2};$$

$$\frac{4x + \sqrt{4x + 48}}{2x + \sqrt{x + 12}} = \frac{2}{3};$$

$$\sqrt{6x^2 - 19x + 7} + \sqrt{3x^2 + 25x + 8} = 3x + 1;$$

$$\frac{x + \sqrt{5 - x}}{x - \sqrt{5 - x}} + \frac{5}{3} = 0;$$

$$\frac{x + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - x} = \frac{\sqrt{1 + a^2} - a}{a + \sqrt{1 - a^2}};$$

$$\frac{x^{\frac{1}{m}} + (a^2 - x^2)^{\frac{1}{2m}}}{x^{\frac{1}{m}} - (a^2 - x^2)^{\frac{1}{2m}}} = \frac{b}{c};$$

$$\frac{(x^2 + bx + c^2)^{\frac{1}{n}} + (x^2 - bx + c^2)^{\frac{1}{m}}}{(x^2 + bx + c^2)^{\frac{1}{n}} - (x^2 - bx + c^2)^{\frac{1}{m}}} = \frac{p^m + q^n}{p^m - q^n};$$

$$\frac{a + x - \sqrt{2ax + x^2}}{a + x + \sqrt{2ax + x^2}} = \frac{\sqrt{2a + x} + \sqrt{x}}{\sqrt{2a + x} - \sqrt{x}};$$

$$\left(x + \frac{1}{x}\right)^{\frac{p}{q}} - 5\left(x + \frac{1}{x}\right)^{\frac{p}{2q}}\left(x - \frac{1}{x}\right)^{\frac{p}{2q}} + 6\left(x - \frac{1}{x}\right)^{\frac{p}{q}} = 0.$$

## PROPERTIES OF THE ROOTS OF QUADRATIC EQUATIONS.

**118.** If the equation

$$x^2+px+q = 0$$

be solved in the usual way, we have

$$x^2+px+\frac{p^2}{4} = \frac{p^2}{4}-q$$

$$\therefore \qquad x+\frac{p}{2} = \pm\sqrt{\frac{p^2}{4}-q},$$

Or, $$x = -\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}.$$

Call these two values of $x$, $x_1$ and $x_2$ respectively,

Then $$x_1 = -\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}$$

$$x_2 = -\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}.$$

By addition we have

$$x_1+x_2 = -p. \qquad (1)$$

By multiplication

$$x_1 x_2 = q. \qquad (2)$$

The above equation has two roots, and it cannot have more than two roots. For, if possible, suppose it to have three roots, $x_1$, $x_2$, $x_3$.

Then, since each of these must satisfy the equation, we have

$$x_1^2+px_1+q = 0$$
$$x_2^2+px_2+q = 0$$
$$x_3^2+px_3+q = 0.$$

Subtracting the second of those equations from the first, we find

$$x_1^2-x_2^2+p(x_1-x_2) = 0$$

Or, $$x_1+x_2+p = 0.$$

Similarly, by subtracting the third from the first, we have

$$x_1 + x_3 + p = 0$$

$$\therefore \quad x_1 + x_2 + p = x_1 + x_3 + p$$

$$\text{And } \therefore \quad x_2 = x_3.$$

Or, the third root is the same as the second; that is, the equation has only two roots.

$$\text{Since} \quad x_1 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$$

$$\text{And} \quad x_2 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q},$$

it is evident that the nature of the roots depends upon the relations of the coefficients $p$ and $q$; and the following conclusions may be drawn.

i. If $\frac{p^2}{4} > q,$

Then $\frac{p^2}{4} - q$ is a positive quantity,

And $\sqrt{\frac{p^2}{4} - q}$ is possible,

$\therefore$ $x_1$ and $x_2$ are both possible.

ii. If $\frac{p^2}{4} = q,$

Then $\frac{p^2}{4} - q = 0$

And
$$\left.\begin{aligned} x_1 &= -\frac{p}{2} \\ x_2 &= -\frac{p}{2} \end{aligned}\right\}$$

Or the two roots are equal.

iii. If $\frac{p^2}{4} < q,$

Then $\frac{p^2}{4} - q$ is negative,

And $\sqrt{\frac{p^2}{4} - q}$ is impossible,

and both the roots are in this case said to be impossible or imaginary equations.

**119.** The above results are collected below, and should *be committed* to memory.

In any quadratic equation of the form $x^2 + px + q = 0$.

1. The sum of the two roots equals the coefficient of the second term with its sign changed.
2. The product of the two roots equals the last term.
3. Every quadratic equation has two, and only two roots.
4. If $p^2 > 4q$; the roots are possible and unequal.
   If $p^2 = 4q$; the roots are possible and equal.
   If $p^2 < 4q$; the roots are impossible and unequal.

By means of these properties the following problems may be easily worked.

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## Examples LVIII.

1. What is the sum and what the product of the roots in the three following equations?

$$x^2 - 3x + 2 = 0$$
$$ax^2 + bx = c$$
$$4(x^2 - a^2) = \left(x - \frac{a}{x}\right)^2.$$

2. One root of each of the following four equations is $(-1)$; find the other root in each case without completing the square.

$$x^2 - 6x - 7 = 0$$
$$\frac{x^2}{4} - \frac{3x}{4} = 1$$
$$7x^2 + 7x = 0$$
$$3x + 4 + \frac{1}{x} = 0.$$

3. State the nature of the roots of the following equations, whether possible or impossible, equal or unequal.

$$x^2 + 4x = -4$$
$$(x+a)^2 + (x-a)^2 = 0$$
$$x^2 + x + 1 = 0$$
$$5x^2 - x = 10$$
$$x(x-a) = 0$$
$$x^2 + x = 72.$$

It was shown in (118) that

$$x_1 + x_2 = -p$$
$$x_1 x_2 = q.$$

Hence the equation $x^2 + px + q = 0$ may be put in the form

$$x^2 - (x_1 + x_2)x + x_1 x_2 = 0$$

Or,
$$(x - x_1)(x - x_2) = 0.$$

If, therefore, the roots of the equation

$$x^2 + px + q = 0$$

be known, the expression $x^2 + px + q$ can be resolved into two simple factors.

Ex. To resolve $x^2 + 3x - 40$ into two simple factors,

Put $x^2 + 3x - 40 = 0$,

Solving this quadratic, we have

$$x = -8$$
$$x = +5.$$
$$\therefore \quad x^2 + 3x - 40 = (x+8)(x-5).$$

(4.) Resolve the following expressions into simple factors.

$$7x^2 - 6x - 640$$
$$x^2 - 42x - 297$$
$$3x^2 - 5x - 12$$
$$4x^2 + 29x + 7$$
$$14x^4 - 29x^2 - 15$$
$$abx^2 - (a^2 - b^2)x - ab$$
$$x^2 + 2ax + a^2 - b^2$$
$$x^4 + 2(a+b)x^3 + (a^2 + 4ab + b^2)x^2 + 2ab(a+b)x + a^2b^2.$$

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## SIMULTANEOUS QUADRATIC EQUATIONS.

**120.** For the same reason as that stated in (70), when there are two unknown quantities, there must be given two independent equations. In the following examples, one or both of the simultaneous equations will be quadratic.

Ex. i. Let

$$\left.\begin{aligned} 3y + x &= 9 \\ 3x^2 - 4xy + y^2 &= 7 \end{aligned}\right\} \quad \begin{aligned} &(1.) \\ &(2.) \end{aligned}$$

be the two equations; the solution will be found as follows:

From (1) $x = 9 - 3y$.

Substitute in (2) this value of $x$;

Then $3(9-3y)^2 - 4y(9-3y) + y^2 = 7$,

Or, $243 - 162y + 27y^2 - 36y + 12y^2 + y^2 = 7$;

$\therefore \quad 40y^2 - 198y = -236$,

Or, $y^2 - \frac{99}{20}y = \frac{-59}{10}$

$$y^2 - \frac{99}{20}y + \left(\frac{99}{40}\right)^2 = \left(\frac{99}{40}\right)^2 - \left(\frac{59}{10}\right)$$

$$= \frac{361}{1600}$$

$$\therefore \qquad y - \frac{99}{40} = \pm\frac{19}{40}$$

$$\therefore \qquad y = \frac{59}{20} \text{ or } 2,$$

are the two values of y.

To find the two values of $x$, substitute for $y$ its value in (1).

Taking the first value of y, $\frac{59}{20}$, we get

$$3 \times \frac{59}{20} - x = 9$$

$$\therefore \qquad x = 9 - \frac{177}{20}$$

$$= \frac{3}{20}.$$

Taking the second value, we have

$$3 \times 2 + x = 9,$$

Or, $x = 3.$

The values therefore are

$$x = 3, \quad y = 2;$$

$$x = \frac{3}{20}, \quad y = 2\frac{19}{20}.$$

---

Ex. ii. Let

$$\left.\begin{aligned} 2x^2 - 9xy + 4y^2 &= 30 \\ x^2 - 3y^2 &= 1 \end{aligned}\right\} \qquad \begin{aligned} &\text{i.} \\ &\text{ii.} \end{aligned}$$

And

be the equations.

Dividing (i.) by (ii.) we have

$$\frac{2x^2 - 9xy + 4y^2}{x^2 - 3y^2} = \frac{30}{1}$$

$$\therefore \qquad 28x^2 + 9xy - 94y^2 = 0.$$

Divide by $28y^2$,

Then $$\left(\frac{x}{y}\right)^2 + \frac{9}{28}\left(\frac{x}{y}\right) = \frac{94}{28},$$

which is a quadratic in $\frac{x}{y}$; the roots of this will be found to be

$$\frac{x}{y} = -2$$

$$\frac{x}{y} = \frac{47}{28},$$

Or, $$\left.\begin{aligned} x &= -2y \\ x &= \frac{47}{28}y \end{aligned}\right\}$$

Substitute for $x$ its value $(-2y)$ in (ii.).

Then $$4y^2 - 3y^2 = 1,$$

Or, $$y^2 = 1,$$

And $$\left.\begin{aligned} y &= \pm 1 \\ x &\mp 2 \end{aligned}\right\}.$$

Also

Next, substitute for $x$ its value $\frac{47}{28}y$ in (ii.).

Then $$\left(\frac{47}{28}y\right)^2 - 3y^2 = 1,$$

Or, $$\frac{143}{784}y^2 = 1,$$

$\therefore$ $$y^2 = -\frac{784}{143}$$

$$y = \pm\sqrt{-\frac{784}{143}}$$

$$x = \pm\frac{47}{28}\sqrt{-\frac{784}{143}}$$

$$= \pm 47\sqrt{-\frac{1}{143}} = \pm\frac{47}{\sqrt{-14}}$$

---

Ex. iii. $$x^2 + y^2 = 34 \quad \text{(i.)}$$
$$x - y = 8. \quad \text{(ii.)}$$

Square (ii.); Then $x^2 - 2xy + y^2 = 64$,

Subtract this from (i.).

Then $$2xy = -30,$$

Add this last to (i.)

$\therefore$ $$x^2 + 2xy + y^2 = 4,$$

And $$x + y = \pm 2.$$

But $$x - y = 8.$$

By addition $$2x = 10 \text{ or } 6,$$

$\therefore$ $$x = 5 \text{ or } 3.$$

By subtraction $$2y = -6 \text{ or } -10,$$

$\therefore$ $$y = -5 \text{ or } -3.$$

**121.** The following general rules should be observed in the solution of simultaneous equations.

(a.) *If one of the equations be a simple equation and the other a quadratic* (as in Ex. i.).

*Express* x *in terms of* y *(or* y *in terms of* x, *if more convenient), from the simple equation, and substitute this value in the quadratic: one of the unknown quantities will then be determined by the solution of a quadratic, and the other can be found* (as in Ex. i.).

(b.) *If both the equations be quadratics, and every term containing the unknowns be of two dimensions* ($x^2$, xy, *or* $y^2$), *as in* Ex. ii.

*Divide one of the equations by the other, and a quadratic in* $\left(\frac{x}{y}\right)$ *will result. Solve this, and then express* x *in terms of* y *(or* y *in terms of* x*), and substitute in one of the equations.*

(c.) *When one is a simple equation, the other a quadratic, and the coefficients of* x *and* y *are equal; also those of* $x^2$ *and* $y^2$ (as in Ex. iii.).

*Square the simple equation, then multiply the result by a number which will make the coefficients of* x *and* y *the same as the coefficients of* $x^2$ *and* $y^2$ *in the other equation; subtract one from the other, a value of* xy *will be found. By combining this with the quadratic, a perfect square may be formed, from which* $x + y$ *or* $x - y$ *will be known. Combine this with the original simple equation by addition and subtraction, and the values of* x *and* y *will be known.*

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## Examples LIX.

$$\left.\begin{aligned} x - y &= 15 \\ \frac{x}{2} &= y^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{2x}{3} &= 9 - \frac{y}{3} \\ 2xy &= 140 \end{aligned}\right\}$$

$$\left.\begin{aligned} 2x + 3y &= 12 \\ x^2 + y^2 &= 55 - 7xy \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + y^2 &= 725 \\ x - y &= 15 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 - y^2 &= 500 \\ x - y &= 10 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + xy &= \frac{459}{2} \\ 3y^2 - xy + \frac{21}{2} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x &= 4 + \frac{x-y}{2} \\ y &= \frac{2x+3y+2}{x+2} \end{aligned}\right\} \qquad \left.\begin{aligned} x^2y^2 + 4xy &= 96 \\ x + y &= 6 \end{aligned}\right\} \qquad \left.\begin{aligned} x + y &= 7 \\ xy &= 12 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 - y^2 &= 619 + \frac{2x^2}{7} \\ x^2 + y^2 &= 59 - \frac{y}{2} \end{aligned}\right\} \qquad \left.\begin{aligned} 4x - 3y &= 0 \\ x^2 + y^2 &= 400 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + y^2 &= 136 \\ x^2 - xy &= 40 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + xy &= 35 \\ xy + 2y^2 &= 2 \end{aligned}\right\} \qquad \left.\begin{aligned} y^2 + xy &= -7 \\ x^2 - xy &= 120 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 - xy + y^2 &= 156 \\ x^2 - y^2 &= 84 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x+y}{x-y} &= 4 \\ xy &= 15 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{4y}{x} - \frac{9x}{y} &= 0 \\ y\,(3x+2) &= 485 - 2x \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + y^2 &= 65 \\ xy &= 28 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + y^2 &= 89 \\ x + y &= 13 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 - y^2 &= 1599 \\ xy &= 40 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{x+y}{x-y} &= a \\ x^2 + y^2 &= b^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} x - y : x + y &:: 1 : 15 \\ x^2 - y^2 : x^2 + y^2 &:: 15 : 113 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{x+y}{x^2+y^2} \\ \frac{x^2}{y^2} - \frac{y^2}{x^2} &= \frac{x-y}{y^2} \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + xy + \frac{x}{y} &= 13 \\ \frac{x + xy}{\frac{x}{y} + x} &= 3 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 36 \\ x + y &= 24 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x^3 - y^3}{(x-y)^3} &= \frac{7}{25} \\ x^2 + y^2 &= 52\,(x+y) \end{aligned}\right\} \qquad \left.\begin{aligned} xy\,(x-y) &= 110 \\ \frac{x^2y^2}{1870} &= \frac{1430}{x^2+y^2} \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} - \frac{y}{x} &= \frac{23}{4} \\ 6\,(x^2 + x) &= 23\,(y^2 + y) \end{aligned}\right\} \qquad \left.\begin{aligned} x^4 + y^4 &= 641 \\ x + y &= 7 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^4 - y^4 &= 2320 \\ x - y &= 4 \end{aligned}\right\} \qquad \left.\begin{aligned} x + y &= 6 \\ x^5 + y^5 &= 1056 \end{aligned}\right\} \qquad \left.\begin{aligned} x^6 - y^6 &= 19 \\ x^2 - y^2 &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^4 + y^4 &= 1 + 2xy + 3x^2y^2 \\ x^3 + y^3 &= x + y - x^2y - xy^2 \end{aligned}\right\} \qquad \left.\begin{aligned} xy + xy^2 &= 12 \\ x + xy^3 &= 13 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^3 + y &= 1 \\ y^3 + x &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + xy + y^2 &= 7 \\ xy - x^2 &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} (x-y)\,(x^2-y^2) &= a \\ (x+y)\,(x^2+y^2) &= b \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{y-x}{y^2+x^2} \\ \frac{x^2}{y^2} - \frac{y^2}{x^2} &= \frac{x+y}{y^2} \end{aligned}\right\} \qquad \left.\begin{aligned} y^4 &= x^2\,(5y - 3x \\ x^2 &= 5x - 3y \end{aligned}\right\} \qquad \left.\begin{aligned} x^{\frac{3}{2}} + y^{\frac{2}{3}} &= 3x \\ x^{\frac{1}{2}} + y^{\frac{1}{3}} &= x \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2+xy+y^2 &= 84 \\ x+\sqrt{xy}+y &= 14 \end{aligned}\right\} \qquad \left.\begin{aligned} x^{\frac{1}{2}}+y^{\frac{1}{2}} &= 4 \\ (x+y)\,(x^{\frac{3}{2}}+y^{\frac{3}{2}}) &= 280 \end{aligned}\right\}$$

$$\left.\begin{aligned} x(x+y)+y(x-y) &= 158 \\ \frac{x+y}{x-y} &= \frac{72y}{7x} \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{x}{y}+\frac{y}{x}-\frac{1}{xy} &= 2 \\ &= \frac{xy\,(xy+1)}{3} \end{aligned}\right\}$$

$$\left.\begin{aligned} 2x+y+\sqrt{2x+y+4} &= 23 \\ 4x^2-6x &= y^2+3y \end{aligned}\right\} \qquad \left.\begin{aligned} x^2+y &= 25 \\ xy &= 36 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{(x+y)^{\frac{1}{3}}}{y}+\frac{(x+y)^{\frac{1}{3}}}{x} &= \frac{64}{63} \\ \frac{(x-y)^{\frac{1}{3}}}{y}-\frac{(x-y)^{\frac{1}{3}}}{x} &= \frac{4}{63} \end{aligned}\right\} \qquad \left.\begin{aligned} x^{\frac{1}{3}}+y^{\frac{1}{3}} &= 3 \\ x+y &= 9 \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x+y+\sqrt{x^2+y^2}}{x+y-\sqrt{x^2+y^2}} &= \frac{a}{2x} \\ \sqrt{\frac{a+x}{a-y}} &= \frac{x}{y} \end{aligned}\right\} \qquad \left.\begin{aligned} x^{\frac{1}{p}}+y^{\frac{1}{p}} &= a^{\frac{1}{p}} \\ xy &= c^2 \end{aligned}\right\}$$

$$\frac{xy}{x+y} = \frac{xz}{x+z} = 1; \quad \frac{yz}{y+3} = 3;$$

$$ax+yz = ay+xz = az+xy = c^2;$$

$$x^2+yz+y^2 = xz = c; \quad z^2+xy = a;$$

$$x^2+y^2+z^2 = x+y+z = x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}} = 1;$$

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## PROBLEMS REQUIRING THE SOLUTION OF QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

**122.** In forming the equation from the given questio, ie Rules given in page 75 must be observed. In the soluon of the equations peculiarities often occur. Some of iese will be discussed in the examples worked out below.

1. The sum of two numbers is 20, and the sum of their cubes is 00; find the numbers.

Let $x$ = one of the numbers,

$\therefore$ $20-x$ = the other number.

And $x^3+(20-x)^3 = 3500$ is the equation to be solved

This becomes $60x^2-1200x = -4500$;

A quadratic, from which we find

$$x = 5 \text{ or } 15.$$

If we take the first value of $x$,

$$\left.\begin{array}{lr} \text{Then} & x = 5 \\ \text{And} & 20 - x = 15 \end{array}\right\}\text{ are the two numbers.}$$

If we take the second value of $x$,

$$\left.\begin{array}{lr} \text{Then} & x = 15 \\ \text{And} & 20 - x = 5 \end{array}\right\}\text{ the same two numbers.}$$

In this case, therefore, the two solutions lead to the same result.

This question may also be solved by two simultaneous equations.

These will be
$$\left.\begin{array}{r} x + y = 20 \\ x^3 + y^3 = 3500 \end{array}\right\}$$

the solution of which is more symmetrical than that just given.

2. I have to walk a distance of 144 miles, and I find that if I increase my speed $1\frac{1}{2}$ miles per hour, I can do the journey in 16 hours less than if I walk at my usual rate: find my usual rate of walking.

Let $x =$ the number of miles per hour I usually walk.

And $x + \frac{3}{2} =$ the increased rate.

Now $\frac{144}{x} =$ the number of hours in which I should do 144 miles at the usual rate.

And $\frac{144}{x + \frac{3}{2}} =$ the time ... at the increased rate.

$\therefore \quad \frac{144}{x} = \frac{144}{x + \frac{3}{2}} + 16$ is the equation to be solved.

Or, $x^2 + \frac{3}{2}x = \frac{27}{3}$.

The values of $x$ are found to be

$$x = 3 \text{ or } -\frac{9}{2}.$$

The value 3 is the only one that applies to this problem; 3 miles per hour is therefore the rate of walking.

The negative value of $x$ may be explained by a reference to the following problem;—

I have to walk a distance of 144 miles, and I find that if I *diminish* my speed $1\frac{1}{2}$ miles per hour, I can do the journey in 16 hours *more* than if I walk at my usual rate: find my usual rate of walking.

If we construct an equation from this question, we shall find it to be

$$\frac{144}{x} = \frac{144}{x - \frac{3}{2}} - 16,$$

Or, $\quad x^2 - \frac{3}{2}x = \frac{27}{2}$;

the roots of which are

$$x = -3 \text{ or } +\frac{9}{2},$$

ch are the same numerically as the roots of the former, but are of osite signs. That is, the negative root of the first equation, with sign changed, is the solution of the second question; that is, $4\frac{1}{2}$ es per hour.

'he pupil should carefully compare the statements in the two prons. We shall return to the subject presently.

. The length of a room is 6 yards more than its breadth, and the ; of a carpet to cover it, at 5*s*. per square yard, is £20 11*s*. 3*d*.: find size of the room.

Let $x$ = the length of the room in yards.
$\therefore$ $x - 6$ = the breadth " "
Again $x(x-6)$ = number of square yards in the carpet.
$5x(x-6)$ = the cost in shillings.
Or, $5x(x-6) = 411\frac{1}{4}$.

This reduces to $x^2 - 6x = \dfrac{329}{4}$

which gives $x = 12\frac{1}{2}$ or $-6\frac{1}{2}$;

Ve must take the positive value of $x$.

Then $x = 12\frac{1}{2}$ the length,
And $x - 6 = 6\frac{1}{2}$ the breadth.

'he negative value may be explained by a reference to the follow- question.

'he breadth of a room is 6 yards *less* than its length, and the cost carpet, &c. (the same as before).

'he equation:—

Let $x$ = the breadth of the room in yards.
Then $x + 6$ = the length " "
And $x(x+6) = \dfrac{329}{4}$ is the equation to be solved,
which gives $x = 6\frac{1}{2}$ or $-12\frac{1}{2}$.

1 this case we see that the negative root of the first question, with ign changed, is the solution of the second question.

A farmer bought one flock of sheep for £40, and another flock of sheep less at 5*s*. a-head more; the whole cost of the larger flock £4 more than the cost of the smaller: find the number of sheep ach flock.

Let $x$ = the number of sheep in the larger flock.
$\therefore$ $x - 4$ = " " lesser "
And $\dfrac{800}{x}$ = price of each sheep of the first flock, in shill.
$\dfrac{720}{x-4}$ = " " second "

$$\begin{aligned} \text{Then} \quad \frac{800}{x} &= \frac{720}{x-4} - 5 \text{ is the equation.} \\ \text{Or, } x^2 + 12x &= 640, \\ \text{which gives } x &= 20 \text{ or } -32. \\ \therefore \quad 20 &= \text{the number of the larger flock} \\ \text{And } 20 - 4 &= 16 \quad \text{,,} \quad \text{smaller ,,} \end{aligned}$$

The value $-32$ bears reference to the following question:—

A farmer bought one flock of sheep for £40, and another flock of four sheep *more*, at 5*s*. a-head *less*; the whole cost of the larger flock was £4 *less* than the cost of the smaller: find the number of sheep in each flock.

$$\begin{aligned} \text{Let} \quad x &= \text{the number of sheep in the smaller flock.} \\ \therefore \quad x - 4 &= \quad \text{,,} \quad \text{,,} \quad \text{larger ,,} \end{aligned}$$

Hence we get the equation

$$\begin{aligned} \frac{800}{x} &= \frac{720}{x+4} + 5, \\ \text{Or, } x^2 - 12x &= 640, \\ \text{Whence} \quad x &= -20, \text{ or } x = 32, \text{ the number in the smaller flock.} \end{aligned}$$

The following principle will serve as a guide to the interpretation of the negative roots of quadratic equations.

State a second question, in which all the positive quantities of the original question are made negative, and all the negative positive. Construct the equation corresponding to this new question, and solve it; then the positive root of this new equation will be equal numerically to the negative root of the original equation, and will be the answer to the second question.

In practice, the following rules for writing down the question to which the negative value refers, will be found useful.

Wherever the original question implies *addition*
the new must imply *subtraction*

| | | | |
|---|---|---|---|
| And thus for | *less* | we must write | *greater*, |
| ,, | *greater* | ,, | *less*, |
| ,, | *loss* | ,, | *gain*, |
| ,, | *time past* | ,, | *time future*, |
| | &c. | | &c. |

according to the explanations in p. 7.

5. Divide a line $a$ feet long into two parts, so that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

$$\begin{aligned} \text{Let} \quad x &= \text{the length of the greater part, in feet.} \\ \therefore \quad a - x &= \quad \text{,,} \quad \text{less} \quad \text{,,} \\ \therefore \quad x^2 &= (a - x)\,a \\ \therefore \quad x^2 + ax &= a^2, \end{aligned}$$

Whence we find $x = -\frac{a}{2} \pm a\frac{\sqrt{5}}{2}$,

The positive root of this $= \frac{a}{2}(\sqrt{5} - 1)$.

Now, $\sqrt{5}$ is a quantity which cannot be exactly found, as the process leads to an endless decimal; we may, however, extract the root correctly to as many places of decimals as we please: to three places it gives 2·236.

$$\therefore \quad \frac{a}{2}(\sqrt{5} - 1) = \frac{a}{2}.(1.236) = a \times .618,$$

and if we mark off on the line $AB$ from $A$, a distance $AC = \frac{618}{1000}$ of $AB$, the point $C$ will be the required point nearly.

Compare this with Euclid, B. II. Prop. ii.; and show, from the construction of that prop., that $AH = \frac{\sqrt{5} - 1}{2}.a$ *exactly;* the geometrical construction being accurate, while the algebraical solution gives only an approximate one.

Also, construct a question to interpret the negative root, and draw a line divided in the manner indicated by that root.

The sum of two numbers is 10, and the sum of their squares is less by 4 than twice their product: find the numbers.

Let $x$ be one of the numbers,

$\therefore \quad 10 - x =$ the other number,

And $x^2 + (10 - x)^2 = 2x(10 - x) - 4$

is the equation to be solved.

It becomes $x^2 - 10x = -26$.

Comp. sq. $x^2 - 10x + 25 = -1$,

Ex. sq. root $x - 5 = \pm\sqrt{-1}$,

Or, $x = 5 \pm \sqrt{-1}$.

Now, $-1$ is a quantity whose square root cannot be extracted, for any quantity whatever when squared must give a *positive* result by the law of signs. $\sqrt{-1}$ is therefore called an impossible quantity; and the solution of the above problem is impossible in real numbers.

The same conclusion may always be drawn whenever the value of $x$ involves the square root of a negative quantity.

It may be shown, for another reason, that the above problem is impossible.

Let $a$ and $b$ be any two numbers.

1° Let $a$ be different from $b$.

Then $(a - b)^2$ is a positive quantity.

Or, $(a-b)^2 > 0$

$\therefore$ $a^2 - 2ab + b^2 > 0$.

Add $2ab$ to both of these unequals.

Then $a^2 + b^2 > 2ab$.

2° Let $a$ be equal to $b$.

Then $(a-b)^2 = 0$

Or, $a^2 - 2ab + b^2 = 0$

$\therefore$ $a^2 + b^2 = 2ab$.

It thus appears that "*the sum of the squares of two numbers can never be less than twice their product.*"

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## Examples LX.

(1.) Divide the number 20 into two parts, such that the greater shall equal the square of the less.

(2.) Find two numbers whose sum is 15, and product 26. Also find two numbers whose sum is 8, and product 13.

(3.) Find the number whose square added to its cube is 16 times the next higher number.

(4.) The difference of two numbers is 6, and the sum of their squares is 116: find the two numbers.

(5.) The sum of two numbers is 12, and their product 36: find the two numbers.

(6.) The sum of two numbers is $a$, and their product $= b^2$: find expressions for the two numbers in terms of $a$ and $b$. Also show from the expressions that the product of two numbers cannot be greater than the square of half their sum.

(7.) i. Divide a line 20 ft. long into two parts, such that the square of one part, together with twice the square of the other part, shall be equal to 300 square feet.

ii. Do the same with a line 10 ft. long; and interpret carefully your results in the latter case.

(8.) There are two square fields, the greater of which is 4 times the less, and the side of the greater is 20 yds. longer than the side of the other: find the area of each field.

(9.) A rectangular looking-glass is 2 ft. longer than it is broad, the plate costs 5*s*. per square foot, and the frame 5*s*. per linear foot, and the whole costs £11 10*s*.: find the dimensions of the plate and frame.

(10.) The length of a rectangular room is 2 ft. more than its breadth, and the height is 2 ft. less than the breadth; the carpet is ¾ of a yd. wide, and costs 3*s*. 6*d*. a yd.; and the paper costs 4*d*. a yd., and is ½ a yd. wide. The whole cost of carpet and paper is £6. 5*s*. 7⅜*d*.: find the dimensions of the room.

(11.) Before going into action a regiment was formed into a solid square, 441 men were killed, and the men left are again formed into a solid square; it is found that the number of men in each side of the square is only $\frac{4}{5}$ of the number before the battle: find the number of men in the regiment at first.

(12.) A number of soldiers are formed into a hollow square of three deep; if 324 men are added to the centre they will form a solid square: find the whole number of men.

(13.) The sum of the two digits of a number is 6, and the number itself is three times the product of the two digits: find the number.

(14.) A number consists of three digits, the first two are alike, and the third is greater by one than either of the others; but if each digit be squared, and a new number be formed with these squares for its digits, the new number is greater by three than the double of the original number: find the original number.

(15.) The sum of the reciprocals of two numbers is $\frac{7}{12}$, and the product of the numbers is 12: find them.

(16.) The hind-wheel of a carriage makes 12 revolutions less than the fore-wheel in going 360 yds., but if the circumference of each wheel be diminished by $\frac{3}{4}$ of a yd., it will make 24 less in the same space: find the circumference of each wheel.

(17). A farmer buys a number of oxen for 300 guineas, and, after losing five of them, sells the rest for 4 guineas a-head more than they cost him, and by so doing he loses 15 guineas: find the number of oxen he bought. Also explain the double value of $x$.

(18.) A man has to perform a journey of 150 miles: on the first day he walks a certain fraction of the whole; on the second day he walks the same fraction of what is left; and there are then 96 miles left; find what fraction of his journey he walked each day.

(19.) A man has to travel a certain distance; when he has gone 20 miles he increases his speed 1 mile per hour: if he had travelled at this increased rate during the whole of his journey he would have arrived 40 minutes earlier, but if he had kept on at his first rate he would have arrived 20 minutes later; find the length of his journey.

(20.) One man can reap a field in 5 days less than another, and if they work together they can do it in 6 days: find in what time each could do it alone. (For the notation to this question see p. 91.) Also explain the double value of $x$.

(21.) There are two rectangular cisterns, in each of which the length is 1 ft. more than the breadth, and the breadth 1 ft. more than the depth; also the greater cistern is 1 ft. deeper than the less, and holds 60 cubic ft. more: find the dimensions of each cistern.

(22.) A man walks to the top of a mountain from the base in 8 hrs.: his rate of walking during the first half of the ascent is 1 mile an hour more than during the last half; he then descends the mountain in 3 hrs. 45 min. by walking at a uniform rate per hour, and this rate is equal to the sum of the distances walked in 1 hr. in his ascent on the first half and on the second half respectively: required the distance of the mountain from the base to the summit.

(23.) The area of a square field exceeds that of a circular field by one acre, and the boundary of the former is 400 yds. longer than that of the latter: find the radius of the circle.

*Note.* The area of a circle = $(\text{radius})^2 \times 3.14159$ nearly.

(24.) There are three square grass-plots. The second contains 27 square yds. more than the first, and the third 2405 more than the second; also it costs five times as much to fence round the two larger plots as it does to fence round the smaller: find the length of the side of each field.

## PROBLEMS REQUIRING THE SOLUTION OF QUADRATIC EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

### 122. Examples worked out;—

1. A rectangular looking-glass is 6 inches longer than it is broad, and it is surrounded by a frame of uniform breadth. Now, the area of the frame is just half that of the glass; but if the frame were only half the breadth that it is, its area would be only $\frac{19}{80}$ of the area of the glass: find the size of the glass and the breadth of the frame.

Let $x$ = the length of the glass in inches.
And $y$ = the breadth of the frame „
Then $x - 6$ = the breadth of the glass „
Now, $x(x-6)$ = the area of the glass in sq. „
And $(x+2y)(x-6+2y)$ = the whole area covered by the frame and glass together.

Hence $(x+2y)(x-6+2y) - x(x-6)$ = area of frame,

$\therefore \quad (x+2y)(x-6+2y) - x(x-6) = \frac{1}{2}x(x-6)$
is the first equation.

If we write $y$ instead of $2y$, we shall have the area of the frame in the second case; the expression then is

$$(x+y)(x-6+y) - x(x-6)$$

And $(x+y)(x-6+y) - x(x-6) = \frac{19}{80}x(x-6)$
is the second equation.

Simplifying these equations, and subtracting one from the other, we get

$$y^2 = \frac{x(x-6)}{80} \qquad (1.)$$

Also, by equating the coefficients of $y^2$ in the two equations, and subtracting, we get

$$y = \frac{9x(6 - x)}{80(3 - x)} \qquad (2.)$$

And these two values of $y$ being put equal to each other, lead to the quadratic

$$x^2 - 6x = 720$$

$\therefore \quad x = 30$ or $-24$

$\therefore$ 30 inches is the length of the glass.

$30 - 6 = 24$ inches is the breadth of the glass.

And $y = 3$ inches the breadth of the frame.

The negative value may be explained as in former cases.

2. *A* and *B* are partners in trade: *A* puts in £300 for 16 months, and *B* puts in a certain capital for 12 months; *B* receives for both principal and profit £260, and the whole gain of both is £180: find the gain per cent. per annum by trading, and the capital invested by *B*.

Let $x$ = the number of pounds invested by *B*.

Let $r$ = the gain on £1 in one year.

Then $x + xr = 260$ (1.)

And $300 \cdot \frac{4}{3}r = 180 - xr$ (2.)

are the two equations to be solved.

These give $x = 236\frac{4}{11}$, *B*'s capital,

And $r = \frac{1}{10}$ the gain on £1 per annum.

*Note.* The gain *per cent.* per annum in this case $= 100r = 10$.

3. A cistern has two supply-pipes, and it is found that the greater fills it in 20 minutes less than the other, and that when both the pipes are running, it is filled in 16 minutes less than when the larger one only is running; also the number of gallons supplied by both pipes in one hour is 45 gallons: find the number of gallons in the cistern, the number of gallons which enter per minute by each pipe.

Let $x$ = the number of gallons supplied per hour by the larger pipe.

And $y$ = the number supplied by the smaller.

And $z$ = the whole number of gallons in the cistern.

Then

$$\frac{z}{y} - \frac{z}{x} = 20 \qquad (1.)$$

$$\frac{z}{x} - \frac{z}{x + y} = 16 \qquad (2.)$$

$$x + y = 45 \qquad (3.)$$

are the three equations to be solved.

Dividing (1) by (2), and simplifying, we get

$$\frac{x^2}{y^2} = \frac{9}{4},$$

And this equation, combined with (3), gives

$$x = 27 \text{ gallons per hour by the larger pipe.}$$
$$y = 18 \quad ,, \quad ,, \quad \text{smaller pipe.}$$

Also $z = 1080 =$ the gallons in the cistern.

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## Examples LXI.

(1.) The sum of the squares of two numbers is 13, and the sum of the numbers is 5: find the numbers.

(2.) The sum of the squares of two numbers is 13, and the difference of the numbers is 5: find the numbers.

(3.) The sum of two numbers is greater than the square root of twice their product by 2, and the sum of their squares is 20: find the numbers.

(4.) There are two numbers whose sum multiplied by the greater is 1080, and their difference multiplied by the less is 182: find the numbers.

(5.) A sum of money is to be divided equally among a certain number of persons; if there had been 3 claimants less each would have had £150 more, and if there had been 6 more each would have had £120 less: required the sum divided and the number of persons. (Simple equation.)

(6.) The area of a square field exceeds that of a circular field by one acre, and the boundary of the former is 400 yds. longer than that of the latter: find the radius of the circle.

(7.) A rectangular field is to be enclosed with a fence 600 yds. long; within the fence it is to be surrounded by a gravel road 7½ yds. wide, and in the centre there is to be 3¾ acres of land: find the length and breadth of the whole inclosure.

(8.) There are two rectangular cisterns, in each of which the length is 1 ft. more than the breadth, and the breadth 1 ft. more than the depth; also the difference in the depth of the two cisterns is 1 ft., and the difference in their solid contents is 60 cubic ft.: find the dimensions of each.

(9.) There are two fields containing together 804 square yds.; one is a square, the other a rectangle; the area af the square is twice that of the rectangle, and the perimeter of the rectangular field is greater than the perimeter of the square one by 26 yds; find the dimensions of each field.

(10.) A number consists of three digits; the first two are alike, and the third is greater by one than either of the others; if each digit be squared, and a new number be formed with the squares for its digits, the new number is greater by 3 than the double of the original number: find the number.

(11.) A and B begin a piece of work, and work together till half the work is done; B then goes off to another job, and A finishes it by himself, and it is found that the work now takes 1 day longer than if they had gone on together. If, however, A had gone away instead of B, the work would have taken 3 days longer than it actually did take: find the time that the work occupied.

(12.) A man and a boy carry the same number of bricks in a day. The man carries 5 more bricks in each load than the boy, but goes six times less in 1 hour than the boy does. The boy begins at 6, and the man at $9\frac{1}{2}$; by 12 o'clock the boy has carried 441 bricks more than the man, and at 1 o'clock half the whole number of bricks is carried; they stop 1 hour, and then the man goes on till $7\frac{1}{2}$, the boy till 7: find the number of bricks carried altogether.

(13.) A man has to perform a certain journey: on the first day he walks a certain fraction of the whole, on the second day he walks the same fraction of what is left, and at the end of the second day he finds he has still 128 miles to walk. Now, if he had walked only half as many miles on the first day as he actually did walk, and had gone on according to the same law as he did in the first case, he would have had 162 miles left at the end of the second day: find the length of his journey, and the number of miles he walked every day.

(14.) A boy spends 11*s*. in apples and oranges, and gets twice as many apples as oranges; another boy spends 8*s*. $1\frac{1}{2}$*d*. at the same shop, but he gets twice as many oranges as apples, and he has 12 more oranges than the former. If oranges be 6*d*. a dozen more than apples, find the number bought by each boy.

(15.) A purse contains *a* pieces of money of two different kinds. Each gold coin is worth as many shillings as there are silver pieces, and each silver coin is worth as many shillings as there are gold pieces. The value of the whole is *b* shillings: find the number of pieces of each kind.

> *Ex.* When the whole number of pieces is 25 and the value of the whole is £10.

(16.) Two farmers go to market with a number of sheep. The first sells 3 more than the second, and the whole sum which they both receive is £36. If the first had sold as many as the second he would have received £15, and if the second had sold as many as the first he would have received £20 5*s*: find the number of sheep sold by each.

(17.) A draper bought a quantity of cloth, intending to sell it in 6 months at 1*s*. 6*d*. per yd. more than he gave for it; but, the price of cloth falling 6*d*. a yd., found that by selling it at that time he should lose the cost-price of 5 yds. He therefore kept it for a year, and then sold it at 1*s*. per yd. under cost price, and found that he lost a sum which was 5*s*. less than the sum he expected to gain: find the number of yards he bought, reckoning interest at 5 per cent.

(18.) A and B are two towns on a river 36 miles apart. A rows down the river from A to B in 8 hrs. by rowing the first half of the distance and walking the rest. In returning he walks the first half at the same rate as before, but, the stream being in his favour, he rows $1\frac{1}{2}$ mile per hr. faster than he did in going, and so does the whole distance in 7 hrs.: find his rate of walking and of rowing.

(19.) Two men, whose rates of walking are 2 miles and 3 miles an hour respectively, start at the same instant from the right angle of a triangular field to walk round it in opposite directions. They first meet at one of the acute angles, and secondly they meet at the distance of 100 yds. from the other acute angle: find the length of the sides of the field.

(20.) A railway train, after travelling $1\frac{1}{2}$ hr., is detained half-an-hour; it then goes on at three-fourths of the former rate, and arrives at the terminus 3 hrs. late. If the stoppage had taken place 30 miles farther on, the train would have arrived half an hour sooner than it did: find the rate per hour of the train.

(21.) A, who can row 2 miles an hour faster than B, sets out from Maidstone to row to Rochester at the same time that B sets out from Rochester to row to Maidstone. A has the tide running 2 miles an hour in his favour until they meet at low water; they rest half-an-hour till the tide turns, and then row on. A gets to Rochester 1 hour before B gets to Maidstone. In the evening they start at the same time to return; A has the tide in his favour the whole way, and does the distance in $4\frac{1}{2}$ hrs. less than B: find the distance from Maidstone to Rochester, and A's and B's rates of rowing.

(22.) A number consists of 3 digits, and it is greater by one than 36 times the sum of the digits; also the middle digit is twice the product of the other two, and is greater by two than their sum; find the number.

(23.) Two merchants engage in a speculation. One invests his capital for 15 months, and receives as his capital and profit £296. The other invests his for 16 months, and receives £175 for his capital and profit. Their united profits are £95: find the rate per cent. per annum profit, also the capitals invested.

(24.) The total length of the edges of a rectangular parallelopiped is 72 ft., its entire surface measures 174 square ft., and its solid content is 110 cubic ft.: find the dimensions of the body.

# CHAPTER XIII.

## RATIO, PROPORTION, AND VARIATION.

### RATIO.

**123.** When two quantities are of the same kind, they may be compared in respect of magnitude.

Thus, if we take two straight lines of 6 ft. and 4 ft. in length respectively, the first is greater than the second, and contains it 1½ times, or is 1½ times as great. Again, if we take two weights of 3 lbs. and 9 lbs., the first is less than the second, and contains it one-third of a time, or is one-third of it.

But quantities cannot be compared unless they be of the same kind; for example, a line 6 ft. long cannot be compared with a day 24 hours long, nor a weight of 10 lbs. with an area of 5 square feet.

**124.** *Def.*—*Ratio* is the relation which one quantity bears to another quantity of the same kind in respect of magnitude.

The *value of the ratio* is the number of times, or parts of a time, that the first contains the second.

If $a$ and $b$ denote the number of units in the two quantities, the ratio of $a$ to $b$ is written thus: $a : b$.

$a$ is called the antecedent of the ratio; $b$ is called the consequent; $a$ and $b$ are called the terms of the ratio.

The value of the ratio $a : b$ is expressed by the fraction $\frac{a}{b}$, for this fraction expresses the number of times that $a$ contains $b$.

**125.** *Def.*—When the antecedent of one ratio is the same multiple or part of its consequent as the antecedent of another ratio is of its consequent, the two ratios are said to be equal.

Thus, the ratio 10 : 4 is equal to the ratio 15 : 6;

For the value of the ratio 10 : 4 is $\frac{10}{4}$ or $\frac{5}{2}$;

And the value of the ratio 15 : 6 is $\frac{15}{6}$ or $\frac{5}{2}$;

Also the ratios $4 : 9$; $4a : 9a$; $12b : 27b$ are all equal.

Hence it is evident that if both the terms of a ratio be multiplied or divided by the same quantity, the value of the ratio is not altered.

**126.** To find which is the greater of two given ratios, we must express the value of each ratio as a fraction, and then compare the two fractions.

**Ex.** Which is the greater ratio, 9 : 5 or 15 : 8?

The values of the ratios are $\frac{9}{5}$ and $\frac{15}{8}$ respectively,

Or, $\frac{72}{50}$ and $\frac{75}{40}$,

Hence 15 : 8 is the greater ratio.

Ex. 2. Which is the greater ratio, $a + 1 : a + 2$ or $a : a + 1$?

The values of the ratios are $\frac{a+1}{a+2}$ and $\frac{a}{a+1}$;

and these when reduced to a common denominator become

$$\frac{a^2 + 2a + 1}{(a+1)(a+2)} \text{ and } \frac{a^2 + 2a}{(a+1)(a+2)};$$

Therefore the former is the greater ratio.

**127.** *Def.*—When the antecedent is *greater than* the consequent, the ratio is said to be one of *greater inequality;* when less, the ratio is said to be one of *less inequality;* when *equal*, the ratio is said to be one of *equality.*

**128.** *Prop.*—A ratio of greater inequality is diminished by adding any positive quantity to both the terms of the ratio.

Let $a : b$ be the ratio of greater inequality,

$\therefore$ $a > b$.

Let $x$ be added to $a$ and $b$,

Then the terms become $(a + x)$ and $(b + x)$.

To find which is the greater ratio, we must compare the two fractions

$$\frac{a+x}{b+x} \text{ and } \frac{a}{b};$$

These, when reduced to a common denominator, become

$$\frac{ab+bx}{b(b+x)} \text{ and } \frac{ab+ax}{b(b+x)}.$$

But since $b < a$

$\therefore$ $bx < ax$

And $ab + bx < ab + ax$,

that is, the value of the new ratio $a + x : b + x$ is less than that of the ratio $a : b$; or the ratio $a : b$ has been diminished by adding an equal quantity to both its terms.

The proposition may be verified by taking any numerical fraction, whose numerator is greater than the denominator, and adding the same quantity to both numerator and denominator.

Take the ratio $4 : 3$.

If 1 be added to both these terms, it becomes

$$5 : 4,$$

And it will be seen that $\frac{5}{4} < \frac{4}{3}$.

**129.** By a similar method to the above, the three following propositions may be proved, and they should all be carefully written out by the student.

(i.) A ratio of less inequality is increased by adding any positive quantity to both terms of the ratio.

(ii.) A ratio of greater inequality is increased by taking from both terms of the ratio any positive quantity which is less than each of those terms.

(iii.) A ratio of less inequality is diminished by taking from both terms of the ratio any positive quantity which is less than each of those terms.

**130.** *Def.*—If the antecedents of any number of ratios be multiplied together, and the consequents be also multiplied together, a new ratio is obtained, which is said to be compounded of all the former ratios.

Ex. Take the three ratios

$$a : x$$
$$b : y$$
$$c : z.$$

Then $abc : xyz$ is called the compounded ratio of the first three ratios.

Similarly, if the two ratios

$$2 : 3$$

And $6 : 8$

be compounded, the result is

$$12 : 24$$

Or, $1 : 2.$

131. *Def.*—When two *equal* ratios are compounded, the result is called the *duplicate ratio* of either of the first ratios.

Thus, if the ratios $a : b$

And $a : b$

be compounded, we have $a^2 : b^2$,

which is called the duplicate ratio of $a : b$.

132. *Def.*—When three *equal* ratios are compounded, the result is called the *triplicate ratio* of either of the others.

Thus $a^3 : b^3$ is called the triplicate ratio of $a : b$.

Similarly, the ratio $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the subduplicate ratio of $a : b$

And $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ the subduplicate ratio of $a : b$.

## Example worked out.

Two vessels each contain a mixture of wine and water: the first has three gallons of wine to five gallons of water; the other, seven of wine and ten of water. In what ratio must the liquor of the two casks be taken so as to make a mixture in which there shall be five gallons of wine to eight gallons of water?

Let $3x$ = No. of galls. of wine taken from the first cask.

$\therefore$ $5x$ = " water " "

Let $7y$ = No. of galls. of wine from the second cask.

$\therefore$ $10y$ = " water " "

Then $3x + 7y$ = No. of gallons of wine in the mixture.

And $5x + 10y$ = " water "

Then $3x + 7y : 5x + 10y = 5 : 8$ by the hypothesis,

$\therefore \quad 25x + 50y = 24x + 56y$

And $x = 6y,$

$\therefore \quad x : y = 1 : 6,$

And $8x : 17y = 8 : 102 = 4 : 51.$

That is, for every four gallons taken from the first cask, there must be fifty-one taken from the second.

## Examples LXII.

1. Express the values of the following ratios in their simplest form:

$2:5$; $14:21$; $80:20$; $75:100$; $2\frac{1}{4}:3\frac{1}{8}$; $\frac{1}{2}:\frac{1}{4}$; $\frac{3}{4}:\frac{3}{8}$; $a:ab$; $a^2:ab$; $a+b:a$; $a^2+ab:a+b$; $x-y:x+y$; $(x-y)^2:x^2-y^2$; $x^2y^2:x^3y+xy^3$; $a^2+b^2:a^4-b^4$; $xy\,(a^2-b^2):ab\,(x^2-y^2)$; $\sqrt{x}+\sqrt{a}:x-a$; $x^{\frac{1}{3}}+a^{\frac{1}{3}}:x+a$;

$$\frac{x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}:\frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}{x^{\frac{1}{3}}-y^{\frac{1}{3}}};\ \left(\frac{1}{1+x}+\frac{x}{1-x}\right)$$

$$:\left(\frac{1}{1-x}-\frac{x}{1+x}\right);\ \sqrt{2}+1:\sqrt{2}-1;\ \left(\frac{1}{2}\right)^{\frac{1}{2}}:\left(\frac{2}{3}\right)^{\frac{2}{3}};$$

$$\left(\frac{1}{2}\right)^{\frac{1}{4}}:\left(\frac{3}{4}\right)^{\frac{1}{2}};\ 3^{\frac{1}{2}}+1:\frac{1}{3^{\frac{1}{2}}-1}.$$

2. Find which is the greater of the following ratios:

$2:5$ or $1:2$; $1:3$ or $2:7$; $5:6$ or $15:16$ or $4:5$; $4\frac{1}{2}:2\frac{1}{4}$ or $2\frac{1}{4}:1\frac{3}{4}$; $\frac{1}{2}:\frac{1}{4}$ or $\frac{3}{4}:\frac{5}{8}$ or $7\frac{1}{2}:3\frac{3}{4}$; $a+b:a-b$ or $3a+4b:3a-2b$; $a+\frac{1}{2}:a-\frac{1}{2}$ or $3a+1:3a-1$; $1:x^2+y^2$ or $x+y:x^3+y^3$; $x^2+xy+y^2:(x+y)^2$ or $(x-y)^2:x^2-xy+y^2$; $\left(\frac{1}{2}\right)^{\frac{1}{2}}:\left(\frac{2}{3}\right)^{\frac{2}{3}}$ or $\left(\frac{1}{2}\right)^{\frac{1}{3}}:\left(\frac{3}{4}\right)^{\frac{1}{2}}$;

$$\frac{x^2-ax+a^2}{(x+a)^2}\ \text{or}\ \frac{(x-a)^2}{x^2+ax+a^2}.$$

3. Find the ratio compounded of the following:

$2:3$, $3:4$, and $4:5$; $1:2$ and $1:3:4:9$ and $9:4$; $a:b$, $b:c$, and $c:d$; $a+x:a-x$, $(a-x)^2:(a+x)^2$, and $(a-3x):(a+3x)$.

4. Write down the duplicate ratios of

$2:3$, $1:10$, $1:\frac{1}{2}$; $\frac{1}{2}:\frac{1}{3}$; $2\frac{1}{4}:4\frac{1}{2}$; $a:c$; $\frac{a}{b}:\frac{b}{c}$; $a+x:a-x$; $a+1:a-1$; $\sqrt{a+x}:\sqrt{a-x}$; $a^{\frac{1}{2}}+b^{\frac{1}{2}}:(a-b)^{\frac{1}{2}}$.

5. Write down the triplicate ratio of

$5:1$; $6\frac{1}{4}:2\frac{1}{2}$; $a-1:a$; $xy:\sqrt{x^2y+xy^2}$; $a^{\frac{1}{3}}:b^{\frac{1}{3}}$; $(a-x)^2:(a+a)^3$; $\left(\sqrt{\frac{x+y}{x-y}}+\sqrt{\frac{x-y}{x+y}}\right)^{\frac{1}{3}}:\left(1-\frac{x+y}{\sqrt{x^2-y^2}}\right)$.

Also the subduplicate ratio of

$16:9$; $1:4$; $\frac{1}{4}:\frac{1}{9}$; $a^2x^2:(a+x)^2$; $a^m x^m\left(\frac{a^m}{x^m}+2+\frac{x^m}{a^m}\right)$; and $x^{2m}-x^{2m}$.

And the subtriplicate ratio of

$$27 : 4;\ \frac{1}{8} : \frac{1}{64};\ (a+x)^3 : 1;\ \left(\frac{a+x}{a-x}\right)^3 : \left(\frac{a-x}{a+x}\right)^3.$$

6. Compound the duplicate and subduplicate ratios of $1 : 4$.

Write down the duplicate, subduplicate, triplicate, and subtriplicate ratios of $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} : \left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)^{\frac{1}{3}}$;

And find the ratio compounded of them all.

7. Two numbers are in the ratio $1 : 3$, and if 7 be added to each the resulting numbers are in the ratio $1 : 2$: find the numbers.

8. In a certain population, the number of births in each year is to the population at the beginning of that year as $1 : m$, and the number of deaths as $1 : n$; in the first year the population is increased by 750, and in the next year by 800: find the population at the beginning of that year.

Take as examples $\left.\begin{array}{l} m = 12 \\ m = 16 \end{array}\right\}$ and explain each of those cases.

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## PROPORTION.

**133.** *Def.*—If there be four quantities, such that the first has the same ratio to the second which the third has to the fourth, then the four quantities are said to be proportionals.

Thus if $a, b, c, d$ be the four quantities, and if

$$\frac{a}{b} = \frac{c}{d},$$

Then $abcd$ are proportionals.

Ex. 2, 4, 7, 14 are proportionals,

For, $$\frac{2}{4} = \frac{1}{2}$$

And $$\frac{7}{14} = \frac{1}{2},$$

Or, the ratio of 2 to 4 is the same as the ratio of 7 to 14.

When $a\,b\,c\,d$ are proportionals, the proportion is thus written,

$$a : b :: c : \mathrm{d},$$

which is read as follows:

*a* is to *b* as *c* is to *d*.

**134.** In order that four quantities may be proportionals, it is not necessary that they should *all* be of the same kind, but it is necessary that the first should be of the same kind as the second, and the third of the same kind as the fourth.

Thus, 6 shillings and 12 shillings have the same ratio to each other as 2 days and 4 days, for the value of each ratio is $\frac{1}{2}$;

$\therefore \quad 6 : 12 :: 2 : 4.$

*Note.* The first and last terms of a proportion are called the extremes; the others are called the means.

**135.** From the above definitions several important properties of proportionals may be deduced. Some of these we now proceed to prove.

i. If four quantities be proportionals, the product of the extremes is equal to the product of the means.

Or if $\quad a : b :: c : d,$

Then shall $\quad ad = bc.$

For since $\quad a : b :: c : d.$

$\therefore \quad \frac{a}{b} = \frac{c}{d}$ by Def.

And $\quad \frac{a}{b} \times bd = \frac{c}{d} \times bd,$

Or, $\quad ad = bc.$

ii. If the product of two quantities be equal to the product of two others, the four quantities shall be proportionals.

Or if $\quad ad = bc,$

Then $\quad a : b :: c : d;$

For since $\quad ad = bc,$

$\therefore \quad \frac{ad}{bd} = \frac{bc}{bd};$

Or, $\quad \frac{a}{b} = \frac{c}{d},$

$\therefore \quad a : b :: c : d.$

The factors of one product form the extremes of the proportion, and the factors of the other the means.

iii. If $\quad a : b :: c : d,$

Then shall $\quad b : a :: d : c$ Invertendo (by inversion),

Since $\quad a : b :: c : d,$

$\therefore \quad \frac{a}{b} = \frac{c}{d},$

And $\quad 1 \div \frac{a}{b} = 1 \div \frac{c}{d},$

Or, $$\frac{b}{a} = \frac{d}{c},$$

$\therefore$ $b : a :: d : c.$

iv. If $a : b :: c : d$ Alternando (by alternation),

Then shall $a : c :: b : d,$

Since $$\frac{a}{b} = \frac{c}{d},$$

$\therefore$ $ad = bc,$ by (i.)

And $$\frac{ad}{cd} = \frac{bc}{cd},$$

Or, $$\frac{a}{c} = \frac{b}{d},$$

$\therefore$ $a : c :: b : d.$

*Observe.* This proposition does not hold unless the four quantities be all of the same kind; for if $a$ and $c$ represent different kinds of quantities, they cannot have a ratio to one another; and the result of this proposition in that case has no meaning.

v. If $a : b :: c : d$ Componendo by (composition),

Then shall $a + b : b :: c + d : d,$

Since $$\frac{a}{b} = \frac{c}{d}.$$

Add 1 to both sides of this equation;

$\therefore$ $$\frac{a}{b} + 1 = \frac{c}{d} + 1;$$

Or, $$\frac{a + b}{b} = \frac{c + d}{d},$$

$\therefore$ $a + b : b :: c + d : d.$

vi. If $a : b :: c : d$ Dividendo (by division),

Then shall $a - b : b :: c - d : d,$

Since $$\frac{a}{b} = \frac{c}{d},$$

Subtract 1 from both sides;

$\therefore$ $$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

Or, $$\frac{a - b}{b} = \frac{c - d}{d},$$

$\therefore$ $a - b : b :: c - d : d.$

vii. If $a : b :: c : d$ Convertendo (by conversion),

Then shall $a : a - b :: c : c - d,$

Since $\frac{a}{b} = \frac{c}{d}$,

$\therefore$ $\frac{b}{a} = \frac{d}{c}$, by iii.

And $1 - \frac{b}{a} = 1 - \frac{d}{c}$,

Or, $\frac{a-b}{a} = \frac{c-d}{c}$,

$\therefore$ $a - b : a :: c - d : c$;

Or, $a : a - b :: c : c - d$ by iii.

The words '*invertendo*,' &c., written in the margin, are the names iven to the corresponding geometrical propositions in Euclid's Fifth ook. The algebraical propositions are sometimes quoted by the ame names.

viii. If $a : b :: c : d$,

Then shall $a + b : a - b :: c + d : c - d$;

Since $\frac{a}{b} = \frac{c}{d}$,

$\therefore$ $\frac{a+b}{b} = \frac{c+d}{d}$, by v.

And $\frac{a-b}{b} = \frac{c-d}{d}$ by vi.

$\therefore$ $\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$.

Or, $\frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d}$

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

$\therefore$ $a + b : a - b :: c + d : c - d$.

The three last propositions are often useful in simplifying propor- ions or equations which contain complicated expressions.

Thus, if $a - 2 : 2 :: b - 3 : 3$,

Then $(a - 2 + 2) : 2 :: (b - 3 + 3) : 3$, by v.

Or, $a : 2 :: b : 3$, a simpler form.

Again, if $a + x : x :: b + y : y$,

Then $(a + x - x) : x :: (b + x - y) : y$, by vi.

Or, $a : x :: b : y$.

Thirdly, if we have the equation

$$\frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \frac{\sqrt{b^2 + y^2} + y}{\sqrt{b^2 + y^2} - y},$$

We get $$\frac{2\sqrt{a^2+x^2}}{2x} = \frac{2\sqrt{b^2+y^2}}{2y}, \quad \text{by viii.}$$

Or, $$\frac{\sqrt{a^2+x^2}}{x} = \frac{\sqrt{b^2+y^2}}{y};$$

Square both sides of this,

$$\therefore \quad \frac{a^2+x^2}{x^2} = \frac{b^2+y^2}{y^2};$$

Or, $$\frac{a^2}{x^2} = \frac{b^2}{y^2}, \quad \text{by vi.}$$

Take square root of both sides,

$$\therefore \quad \frac{a}{x} = \frac{b}{y}.$$

Therefore we may use this simple form instead of the complica expressions given at first.

ix. If $a : b :: c : d$

And $c : d :: e : f$,

Then shall $a : b :: e : f$;

For $$\frac{a}{b} = \frac{c}{d}$$

And $$\frac{c}{d} = \frac{e}{f}$$

$$\therefore \quad \frac{a}{b} = \frac{e}{f};$$

Or, $a : b ;; e : f$.

x. If there be three quantities, $a$, $b$, $c$, and three others $d$, $e$, $f$, s that

$a : b :: d : e$

And $b : c :: e : f$,

Then shall $a : c :: d : f$, Ex Æquali (from equa of ratios).

For $$\frac{a}{b} = \frac{d}{e}$$

And $$\frac{b}{c} = \frac{e}{f}.$$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{d}{e} \times \frac{e}{f};$$

Or, $$\frac{a}{c} = \frac{d}{f};$$

$\therefore \quad a : c :: d : f$.

This proposition may be extended as follows:—

If there be any number of quantities more than two, and as many others, such that the first is to the second of the first set as the first is to the second of the second set, and the second is to the third of the first set as the second is to the third of the others, and so on to the end; then shall the first be to the last of the first set as the first is to the last of the second set.

Let $a, b, c, d \ldots\ldots h, k, l$ be the first set of quantities,
And $m, n, p, q \ldots\ldots r, s, t$ be the second set "

And let $a : b :: m : n$

$$b : c :: n : p$$

&c. &c.

$$k : l :: s : t$$

Then shall $a : l :: m : t,$

For $\frac{a}{b} = \frac{m}{n};$

Also

$$\frac{b}{c} = \frac{n}{p}$$

$$\frac{c}{d} = \frac{p}{q}$$

&c. &c.

$$\frac{h}{k} = \frac{r}{s}$$

$$\frac{k}{l} = \frac{s}{t}.$$

Multiplying both sides of all these equations together, we have

$$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \ldots \times \frac{h}{k} \times \frac{k}{l} = \frac{m}{n} \times \frac{n}{p} \times \frac{p}{q} \times \ldots \times \frac{r}{s} \times \frac{s}{t}.$$

Now it is evident that on each side all the numerators except the first are cancelled, and all the denominators except the last. Hence there remains

$$\frac{a}{l} = \frac{m}{t},$$

Or, $a : l :: m : t.$

xi. When any number of magnitudes are proportionals, as any one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $a : b :: c : d :: e : f,$

Then shall $a : b :: a + c + e : b + d + f;$

For $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{b} = \frac{e}{f}.$

$\therefore$ $ad = bc;$

Also $af = be$

And $ab = ba.$

By addition,

$$ab + ad + af = ba + bc + be;$$

Or, $a(b + d + f) = b(a + c + e),$

$\therefore$ $\dfrac{a}{b} = \dfrac{a + c + e}{b + d + f}$

And $a : b :: a + c + e : 5 + d + f.$

In the same way the proposition might be proved for any larger number, by putting the fraction $\frac{a}{b}$ equal to all the other ratios in succession, and clearing of fractions, and adding the resulting equations together.

**136.** To find a fourth proportional to three given quantities, $a$, $b$, $c$, is to find a fourth quantity, such that

$$a : b :: c : \text{the fourth quantity}.$$

To find a third proportional to two given quantities, $a$, $b$, is to find a third quantity, such that

$$a : b :: b : \text{the third quantity}.$$

To find a mean proportional between two given quantities, $a$, $b$, is to find a mean such that

$$a : \text{mean} :: \text{mean} : b.$$

### Examples worked out.

1. If $a : b :: b : c,$

Then shall $a : c :: a^2 : b^2,$

For $\dfrac{a}{b} = \dfrac{b}{c};$

$\therefore$ $\dfrac{a}{b} \times \dfrac{b}{c} = \dfrac{a}{b} \times \dfrac{a}{b},$

Or, $\dfrac{a}{c} = \dfrac{a^2}{b^2};$

$\therefore$ $a : c :: a^2 : b^2.$

Similarly, if $a : b :: b : c :: c : d,$

Then shall $a : d :: a^3 : b^3;$

For $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d};$

$$\therefore \qquad \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b};$$

Or,

$$\frac{a}{d} = \frac{a^3}{b^3}.$$

$$\therefore \qquad a : d :: a^3 : b^3.$$

Similar relations may be proved whenever there is a number of equal ratios, such that the consequent of any one of the ratios is the antecedent of the preceding.

2. Simplify the proportion

$$a+b+c+d : a+b-c-d :: a-b+c-d : a-b-c+d.$$

First we have $a+b : c+d :: a-b : c-d$ (by viii),

And $a+b : a-b :: c+d : c-d$ (by iv);

$\therefore$ $a : b :: c : d$ (by viii).

If $a : b :: c : d :: e : f$; show that

$$\frac{a+b+c+d+e+f}{a-b+c-d+e-f} = \frac{a+b}{a-b} = \frac{c+d}{c-d} = \frac{e+f}{e-f};$$

Since $\frac{a}{b} = \frac{e}{f} \therefore \frac{a+b}{a-b} = \frac{e+f}{e-f}$ by (viii.).

Also $\frac{c}{d} = \frac{e}{f} \therefore \frac{c+d}{c-d} = \frac{e+f}{e-f}$;

Hence we have

$$a+b = (a-b)\frac{e+f}{e-f}$$

$$c+d = (c-d)\frac{e+f}{e-f}$$

$$e+f = (e-f)\frac{e+f}{e-f}.$$

By addition,

$$a+b+c+d+e+f = (a-b+c-d+e-f)\frac{e+f}{e-f};$$

$$\therefore \frac{a+b+c+d+e+f}{a-b+c-d+e-f} = \frac{e+f}{e-f} \text{ by division.}$$

$$\left.\begin{aligned} &= \frac{c+d}{c-d} \\ &= \frac{a+b}{a-b} \end{aligned}\right\} \text{ by the above.}$$

3. Gunpowder is composed of nitre, sulphur, and charcoal; and it is known that in any given quantity of gunpowder the weight of the nitre and the sulphur together is to the weight of the nitre and the charcoal together as 17 : 18; also the weight of the nitre and the sulphur together is to the weight of the sulphur and charcoal together

as 17 : 5. Find the proportions in which the three ingredients are combined.

Let $x$ = number of lbs. of nitre in the gunpowder.
$y$ = „ „ sulphur „
$z$ = „ „ charcoal „

Then $x+y : x+z :: 17 : 18$ (1)
And $x+y : y+z :: 17 : 5$ (2)
are the two proportions.

From these we get the two equations

$$x + 18y = 172$$

And

$$5x - 12y = 172,$$

$$\therefore \quad 30y = 4x;$$

Or,

$$\frac{y}{x} = \frac{2}{15}.$$

Similarly we find

$$\frac{z}{x} = \frac{1}{5}.$$

Hence

$$1 + \frac{y}{x} + \frac{z}{x} = 1 + \frac{2}{15} + \frac{1}{5};$$

Or,

$$\frac{x+y+z}{x} = \frac{4}{3}.$$

$$\therefore \quad x : x+y+z :: 3 : 4.$$

Or, the weight of nitre : the whole weight :: 3 : 4.

Similarly the ratios $y : x+y+z$
And $z : x+y+z$ } may be found.

These give the value of the other ingredients.

4. An express train from London meets a parliamentary train from Southampton at 21 miles from London. They arrive at Southampton and London respectively at the same time. The time the express was on the road up to the time of meeting was to the time of the other as 9 : 64. Find the distance from London to Southampton.

Let $r$ = rate of the express train in miles per hour.
$r'$ = „ other „ „
$x$ = distance in miles from Southampton to the place of meeting.

$$\therefore \quad \frac{21}{r'} = \frac{x}{r} \qquad (1.)$$

Also,

$$\frac{x}{r'} : \frac{21}{r} :: 64 : 9 \qquad (2.)$$

From (1.)

$$\frac{r}{r'} = \frac{x}{21},$$

From (2.)

$$\frac{rx}{21r'} = \frac{64}{9}.$$

Dividing one of these equations by the other, we get

$$\left(\frac{x}{21}\right)^2 = \frac{64}{9},$$

$$\therefore \qquad \frac{x}{21} = \frac{8}{3};$$

Or, $$x = 56,$$

And $$x + 21 = 77, \text{ the distance required.}$$

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## Examples LXIII.

1. State the following equations in the form of a proportion:—
$xy = 4a$; $x^2y^2 = \frac{4}{9}$; $a^2 - b^2 = x^2 - 1$; $a^3 - 1 = b^3 + 1$; $x^2 + 5x + 4 = y^2 - 5y + 4$.

2. Apply '*Inversion*' to the following proportions,
$$a : 1 :: b : 2; \quad \frac{1}{a} : \frac{1}{b} :: x : y.$$
Apply '*Componendo*' to the following,
$$x : x - a :: b : y - b; \quad a^2 - x^2 : x^2 :: b^2 - y^2 : y^2.$$
Apply '*Dividendo*' to the following,
$$1 : 1 - x :: 3 : 3 - y; \quad a^2 + x^2 : x^2 :: b + y : y.$$

3. If $a : b :: c : d$, prove the following relations,
$8a - b : 2a + 3b :: 8c - d : 2c + 3d$;
$pa + qb : b :: pc + qd : d$; $a + c : b + d :: ma - nc : mb - nd$;
$a + b + c + d : b + d :: c + d : d$; $(ma + b)^2 : (mc + d)^2$
$:: a^2 - n^2b^2 : c^2 - n^2d^2$; $(a - b)^2 : (c - d)^2 :: ab : cd$;
$a^2 + b^2 : c^2 + d^2 :: \dfrac{a^4}{a^2 + b^2} : \dfrac{c^4}{c^2 + d^2}$; $(a + d) - (b + c)$
$= \dfrac{(c - d)(b - d)}{d}$; $\left(\dfrac{1}{a} - \dfrac{1}{b}\right) - \left(\dfrac{1}{c} - \dfrac{1}{d}\right) = \dfrac{(b - d)(c - d)}{ad^2}$
$= \dfrac{(a - b)(a - c)}{abc}$.

4. If $ad = bc$, then shall
$a + b : c + d :: b : d$; $a + b : a :: c + d : c$;
$a - b : c - d :: a : c$; $ma + nc : mb + nd :: c : d$;

5. If $a : b :: b : c :: c : d :: d : e$,
Then shall $a : e :: a^4 : b^4 :: a^2c^2 :: b^2d^2$;
If $a : b :: c : d :: e : f$,
Then shall $a + c + e : a :: b + d + f : b$;
If $a : b :: c : d$, and $c : e :: e : d$,
Then shall $a : b :: c^2 : e^2$;

If $a - b : c - d :: 1 + ab : 1 + cd$,
Then shall $a + d : b + c :: 1 - ad : 1 - bc$;

If $a : r :: a' : r'$, and $a : r + x :: a' : r' + x$;
Then shall $x : x' :: a : a'$;

If $a : b :: c : d$. and $e : f :: g : h$;
Then shall $ae : bf :: cg : dh$.

6. If $a$ be the greatest and $d$ the least of the four proportionals $a$, $b$, $c$, $d$, show that $a + d > b + c$, and $a^n + d^n > b^n + c^n$.

7. Find two numbers in the ratio of $2 : 5$, and whose difference is to the difference of their squares as $1 : 42$.

8. $a$, $b$, $c$, $d$, be any four given quantities, find a quantity which, when added to each of them, will make the four resulting quantities proportionals; and show that when $a$, $b$, $c$, $d$, are already proportionals, no such quantity can be found.

9. If $y + z : y - z :: (x + y)^2 : (x - y)^2$,
Then shall $x^2 z = y^2 (2x - z)$.

10. If $pq = rs$, and $qt = su$,
Then shall $p : r :: t : u$;

And if $p$, $u$, be the sides of one rectangular field, and $r$, $t$, those of another, show that their areas are equal.

11. If $\dfrac{ax + by}{cx + dy}$ always have the same value, whatever be the values given to $x$ and $y$, then shall $\dfrac{a}{b} = \dfrac{c}{d}$.

12. If $ab(c + d - e - f) + cd(e + f - a - b) + ef(a + b - c - d) = 0$, show that $a - c : a - d :: (b - d)(c - e)(c - f) : (b - c)(d - e)(d - f)$.

13. The cost of making a railway is 1,000,000 rupees, and the expense is to be borne by five towns, of 10,000, 20,000, 30,000, 40,000, and 50,000 inhabitants respectively, in proportion to their populations: find the sum which each town ought to pay.

14. A boatman can row a certain distance down the stream in one hour and a half, but it takes him two hours and a half to row up the same distance: show that he rows his boat four times as fast as the stream alone can carry it.

15. If $\left(\dfrac{1}{n}\right)^{th}$ part of a cask of wine be drawn off, and the cask be filled up with water, and $\left(\dfrac{1}{n}\right)^{th}$ part of the whole be again drawn off, and this be continued $m$ times: find how much wine will be left in the cask after $m$ drawings.

## VARIATION.

137. In considering the relations of algebraical quantities, it is often convenient to regard some as *variables* and others as *constants*. For instance, in the equations

$$ax - by = 0\,;\ 2x - 3y = 0,$$

$x$ and $y$ may receive any different values which are consistent with the equations, and are therefore called the variables of the expressions; but the coefficients 2 and 3 in the one equation, and $a$ and $b$ in the other, preserve the same fixed values, and are called the *constants* of the expressions. This will be seen more clearly if we give to $x$ and $y$ in each of the equations some of the different values which they may receive.

The first equation $2x - 3y = 0$.

is satisfied by the following sets of values of $x$ and $y$,

$$\left.\begin{matrix} x = 3 \\ y = 2 \end{matrix}\right\} \quad \left.\begin{matrix} x = 6 \\ y = 4 \end{matrix}\right\} \quad \left.\begin{matrix} x = 9 \\ y = 6 \end{matrix}\right\} \quad \left.\begin{matrix} x = 12 \\ y = 8 \end{matrix}\right\}$$

as may be seen by trial; that is, the values of $x$ and $y$ may vary, while the coefficients 2 and 3 remain fixed.

The second equation $ax - by = 0$

is satisfied by the following sets of values of $x$ and $y$,

$$\left.\begin{matrix} x = b \\ y = a \end{matrix}\right\} \quad \left.\begin{matrix} x = 2b \\ y = 2a \end{matrix}\right\} \quad \left.\begin{matrix} x = 3b \\ y = 3a \end{matrix}\right\} \quad \begin{matrix} \&c. \\ \&c. \end{matrix}$$

so that $x$ and $y$ are the *variables*, $a$ and $b$ the *constants* of the expression $ax - by$.

138. We may also give a geometrical illustration of the meaning of variables and constants. If we construct a number of different rectangles all on the same base, the altitude and area are the variables, while the base is constant; and, as may easily be shown, the number of square units in the area of one of these rectangles is equal to the product of the number of lineal units in its base and the number of lineal units in its altitude.

Let $A$ denote the number of square units in the area.
$b$ „ „ lineal „ base.
$h$ „ „ lineal „ altitude.

Then $A = bh$.

To this equation, $b$ is the constant; $A$ and $h$ are the variables.

It is to be noticed that the variables which occur in the same expression generally depend upon each other, so that the changes of one of the variables bear some relation to those of the other variables. The nature of these relations will be treated of in the following propositions of this Chapter. The variables which depend upon each other need not be of the same kind: thus, in the example just given, the variables are of different kinds, one being an *area*, the other a *line*; and when it is said that the area equals the line multiplied into a constant quantity, it is only to be understood that the *number of units* in the area equals the *number of units* in the height multiplied into a certain fixed number.

Again, if a man walk at the rate of three miles an hour, the time during which he walks and the distance which he walks in that time are the variables, and 3 is the constant; and it is evident that

$$\text{Distance} = 3 \times \text{Time}.$$

if we understand by "Distance" the number of miles, and by "Time" the number of hours.

**139.** *Def.* 1.—One variable quantity is said to vary directly as another, when the two quantities depend upon each other in such a manner that the number of units in the one always bears the same ratio to the number of units in the other.

Ex. $5x$ varies directly as $x$.

For $$5x : x :: 5 : 1,$$

Or, the ratio of $5x : x$ always equals 5, whatever $x$ may be.

Instead of the words *varies directly as*, or, *varies as*, the symbol $\propto$ is often used, and

$$5x \propto x$$

is read thus $5x$ varies as $x$;

Or, $5x$ varies directly as $x$.

Ex. 2. If the base of a rectangle be a given line, and the altitude and area both vary,

Then $$\text{Area} \propto \text{altitude}.$$

For, as was shown in (138),

$$A = bh,$$

$$\therefore \qquad A : h :: b : 1;$$

And therefore, by definition,

$$A \propto h.$$

*Def.* 2.—When two variable quantities are such that the product of the number of units in the two quantities is constant, one of the variables is said to vary inversely as the other.

Ex. If $x$, $y$ be the number of units in each of the two variables, and $xy = c$ a constant, then $x$ varies inversely as $y$.

Since $$xy = c,$$

$$\therefore \qquad x = c \cdot \left(\frac{1}{y}\right);$$

Or, $$x = \left(\frac{1}{y}\right) \times \text{a constant.}$$

Or, $$x \propto \left(\frac{1}{y}\right);$$

That is $x$ varies as the reciprocal of $y$.

It is therefore the same thing, whether we say

That $x$ varies inversely as $y$,

Or that $x$ varies as the reciprocal of $y$.

Ex. 1. If the area of a rectangle be constant, the base varies inversely as the altitude, for, using the same notation as before,

$$bh = A, \text{ a constant.}$$

Or, $$b = A \cdot \left(\frac{1}{h}\right).$$

Ex. 2. If a given sum of money, as £5, be distributed every day in charity in equal sums, the number of persons relieved will vary inversely as the share of each person.

*Def.* 3.—If the number of units in one variable quantity bear a fixed ratio to the product of the number of units in two other variable quantities, the first variable is said to vary jointly as the other two.

Ex. 1. The area of a triangle varies jointly as the base and perpendicular.

For $$A = \frac{bh}{2},$$

Or, $$\frac{A}{bh} = \frac{1}{2}, \text{ a constant.}$$

Ex. 2. The amount of money paid in weekly wages to a number of men varies jointly as the number of men and the wages of each man.

*Def.* 4.—If the number of units in one variable quantity bear a fixed ratio to the product of the number of units in any number of others, the first variable is said to vary jointly as all these.

The expense of digging a ditch will vary jointly as the length, the breadth, the depth, and the expense of digging one cubic yard.

This is an example of one quantity varying jointly as four others.

*Def.* 5.—If the number of units in one variable quantity bear a fixed ratio to the product of the number of units in a second variable, and the number in the reciprocal of the third, the first is said to vary directly as the second and inversely as the third.

Ex. 1. The base of a triangle varies directly as the area and inversely as the altitude.

For $$A = \frac{bh}{2};$$

Or, $$2A\left(\frac{1}{h}\right) = b;$$

$$\therefore \quad \frac{b}{A\left(\frac{1}{h}\right)} = 2, \text{ a constant quantity.}$$

Ex. 2. The time of performing a journey will vary directly as the length of the journey, and inversely as the rate of walking.

**140.** *Prop.*—If one variable quantity $x$ depend upon two other variables $y$ and $z$, and if when $y$ is constant $x$ vary as $z$, and if when $z$ is constant $x$ vary as $y$; when both $y$ and $z$ vary, $x$ varies as their product.

Or, $$x \propto yz.$$

First, when $y$ and $z$ have the particular values $y_1$ and $z_1$, let the corresponding value of $x = my_1z_1$.

Let $y$ retain the fixed value $y_1$, while $z$ changes from $z_1$ to other values.

Then by hypothesis $x = my_1 . z$, $m$ being constant.

Let $z_2$ be any one of the values which $z$ takes, while $y$ has the constant value $y_1$.

So that $$x = my_1z_2 = mz_2y_1.$$

Let $z$ retain the fixed value $z_2$, while $y_1$ changes to other values.

Then by hypothesis $\quad x = mz_2y.$

And if $y_2$ be any one of the values which $y$ takes,

Then $\quad x = mz_2y_2.$

Now it is evident that $y_2$ and $z_2$ may be any of the values which $x$ and $y$ assume in the course of variation from the original values $y_1$, $z_1$ : that is, when $x$ and $y$ have both varied, the relation between the different values of $xyz$ is such that

$$\frac{x}{yz} = m, \text{ a constant,}$$

Or, $\quad x \propto yz$, when $yz$ are both variable.

Ex. The expense of carriage of goods by railway depends upon the weight carried and upon the distance it is carried.

When the weight is constant and the distance varies, the expense varies as the distance; when the distance is constant and the weight varies, the expense varies as the weight.

When both weight and distance vary, the expense varies as the weight × distance.

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## Examples worked out.

1. If $y \propto x + z$, and if 2, 3, 4 be simultaneous values of $xyz$ respectively, find the relation between $xyz$.

Since $\quad y \propto x + z$, we must have

$y = c(x + z)$, where $c$ is some constant to be determined.

Now, giving $xyz$ the values 2, 3, 4, we have

$$3 = c(2 + 4),$$

Or, $\quad 3 = 6c.$

$\therefore \quad c = \frac{1}{2}$

And $\quad y = \frac{1}{2}(x + z)$ is the required relation.

2. If the square of $y$ vary as the square root of the cube of $x$, and if $x = 4$ when $y = 6$; find the value of $x$, when $y = \frac{3}{4}$.

Here we may assume

$$y^2 = cx^{\frac{3}{2}}, \text{ where } c \text{ is the constant.}$$

$\therefore \quad 36 = 8c,$

Or, $\quad c = \dfrac{9}{2};$

$\therefore \quad y^2 = \dfrac{9}{2}x^{\frac{3}{2}}.$

When $y = \frac{3}{4}$ $$x^{\frac{3}{2}} = \frac{2}{9} \times \left(\frac{3}{4}\right)^2$$
$$= \frac{1}{8};$$
∴ $$x^3 = \frac{1}{64}$$
And $$x = \frac{1}{4}.$$

3. If $(x + y) \propto (x - y)$, shew that $x \propto y$.

Let $(x + y) = c'x - y)$, where $c$ is constant.

∴ $x(c - 1) = y(c + 1)$;

Or, $\frac{x}{y} = \frac{c+1}{c-1}$, which is a constant;

∴ $x \propto y$.

4. The quantity of light passing through an aperture varies directly as the size of the aperture and inversely as the square of the perpendicular distance of the source of light from it. One room is lighted by a window 6 feet long and 4 feet broad, and has a gas lamp 30 feet in front of it. Another room has a window 4½ feet long and 4 feet broad, and has a similar gas lamp 18 feet in front of it. Find which window admits the greatest amount of light.

Let $I_1$ = the amount of light through the first window,
$I_2$ = „ „ second „

$$\therefore \left.\begin{aligned} I_1 &= c.\frac{6 \times 4}{(30)^2} \\ I_2 &= c.\frac{4 \times 4\frac{1}{2}}{(18)^2} \end{aligned}\right\} \text{Where } c \text{ is constant.}$$

$$\therefore \frac{I_1}{I_2} = \frac{6 \times 4}{(30)^2} \times \frac{(18)^2}{18}$$
$$= \frac{12}{25}$$

∴ $I_1 : I_2 :: 12 : 25$;

Or the second has the greatest amount of light.

## Examples LXIV.

1. If $x \propto y$, and $x = 1$ when $y = 2$, find the relation between $x$ and $y$.

2. If $y^2 \propto x$, and $x = 6$, when $y = 3$, find the equation connecting $x$ and $y$.

3. If $y + 2x = 8x - y$, show that $x$ varies as $y$.

4. If $z \propto x + y$, and if when $x = 1, y = 2, z = 4$, find the relation between $xyz$.

5. If $x \propto y$, then $x^2 + y^2 \propto xy$.

6. If $x$ vary as $\left(1 - \frac{y}{b}\right)$, and $\frac{1}{2}a, \frac{1}{2}b$ are simultaneous values of $x, y$ respectively, find the equation between $x$ and $y$.

7. If $x^2 + y^2 \propto z^2$, and if when $x^2 + y^2 = a^2, z = c$, find the relation between $xyz$.

8. If $y \propto x^n$, and if when $x = 1, y = 2$, and when $x = 4, y = 32$, find the value of $n$.

9. If $a^x$ vary as $b^y$, and if $x = 3$ when $y = 2$, show that $a^{x-3} = b^{y-2}$.

10. If $x^2y + xy^2 \propto (x + y)^3$, show that $x \propto y$.

11. If the lengths of two variable lines bear to each other a constant ratio, then the sum of the squares of the two lines varies as the rectangle contained by the two lines.

12. If the sum of two variables vary as the difference of their squares, then the difference of the two quantities is constant.

13. If the sum of the squares of two variables vary as their product, then one of the quantities varies as the other.

14. If $x \propto y$ and $y^2 \propto vz$, then shall $xy \propto vz$.

15. If $U = X + Y$, and $X \propto x$, and $Y \propto \frac{1}{x}$, and if $U = \frac{5}{2}$ when $x = 5$; $U = 2\frac{9}{10}$ when $x = 1$; show that $U = \frac{2}{5}x + \frac{5}{2x}$.

16. If $x + y \propto x - y$, show that $x^2 + y^2 \propto xy$, and that $x^3 + y^3 \propto xy(x \pm y)$.

17. If $x$ vary as $y$, and $y$ vary as $z$, show that $mx + ny + pz$ varies as $a\sqrt{yz} + b\sqrt{xz} + c\sqrt{xy}$, where $m, n, p, a, b, c$, are constants.

18. If $z$ vary jointly as $x$ and $y$, and if when $z = 1, x = 2, y = 3$, find the value of $x$, when $z = 4, y = 1$.

19. If $z$ vary as $x - \frac{1}{x}$, and if when $x = 2, z = 5$, state the relation between $x$ and $y$ in the form of a proportion.

20. Given that the area of a circle varies as the square of its radius, find the ratio between the areas of two circles whose radii are 2 feet and ½ a foot respectively.

21. If a circular plate of metal of 1 foot diameter and 1 inch in thickness be melted down, and cast into another circular plate of one quarter of an inch in thickness, find the radius of the new plate; having given that the volume of a cylinder varies jointly as the square of the radius of its base, and its height.

22. The areas of similar triangles are in the duplicate ratio of their homologous sides; compare the areas of two equilateral triangles, whose sides are 6 feet and 7 feet respectively.

23. The velocity acquired by a body falling from rest varies as the time of its falling from rest; and the velocity of the body at the end of 5 seconds from rest is 161 feet per second; what was its velocity at the end of the first second? Also find how long it must have been falling to acquire a velocity of one mile per minute.

24. The space fallen through by a body from rest varies as the square of the time of its falling. It is found to be 4 seconds falling from a height of 257·6 feet. Hence find the time a stone would be in falling to the bottom of a well 65 feet deep. Also find the relation between the space and time generally.

25. The volume of globe varies as the cube of its radius. If two cannon balls of 3-inch and 4-inch radius respectively be melted down and cast into a single ball, find its radius. Also if the radius of a nine pounder be 2 inches, find the weight of a ball whose radius is 7 inches.

26. The areas of circular sectors vary as the squares of their radii, and the number of degrees in their angles jointly; hence, if the area of a sector of a circle whose radius is 10 feet, and whose angle is 45°, be 39·27 square feet, find the radius of the circle, the area of a quadrant of which is 3·1416 square feet. Also find the area of a circle whose radius is 6 feet.

27. Given that the speed of a train in miles per hour equals the number of horse-powers in the engine diminished by a quantity which varies as the square of the speed, and that when the engine is of 32 horse-power the speed is 30 miles per hour, find what must be the horse-power of the engine when the speed is 20 miles per hour.

# CHAPTER XIV.

## ARITHMETIC, GEOMETRIC, AND HARMONIC PROGRESSION.

**141.** *Def.*—An "*Arithmetic Progression*" is a series of terms more than two, which go on increasing or decreasing by a common difference.

Thus $1, 5, 9, 13$, &c, is an Arithmetic progression.
Also $a, a + b, a + 2b, a + 3b$, &c. " "
*And* $x - 3y, x - 4y, x - 5y$, &c. " "

*Note.* The difference may be a number positive or negative, as 4 in the 1st of the above; or it may be any algebraical quantity, positive or negative, as $b$ in the 2nd series, and $-y$ in the last.

Also, it is clear that the difference between two adjacent terms is the same from whatever part of the series the two terms be taken.

**142.** *Def.*—The quantity which begins the series is called the "*First Term,*" the difference between two adjacent terms is called the "*Common Difference,*" and the amount of all the terms added together is called the "*Sum*" of the terms.

**143.** *Def.*—An "*Arithmetical Mean*" between two given quantities is one whose difference from each of them is the same.

Thus the Arithmetical mean between 10 and 16 is 13; that between $a-b$ and $a+5b$ is $a+2b$, &c.

All the terms of an Arithmetic Progression, except the first and last are called "Arithmetic means" between the first and last.

Thus, 5, 7, and 9, are said to be 3 Arithmetic means between 3 and 11.

**144.** In investigating the properties of these series, the use of general symbols is necessary. We shall first consider a particular case, and then proceed to the more general expression.

Take the series 5, 7, 9, 11, &c.

Now, it is evident from the construction of this, that

$$\begin{array}{rcccl} \text{The 1st term} & = & 5 & = & 5 \\ \text{2nd} \;\;,, & = & 5+1\times 2 & = & 7 \\ \text{3rd} \;\;,, & = & 5+2\times 2 & = & 9 \\ \text{4th} \;\;,, & = & 5+3\times 2 & = & 11 \\ \text{5th} \;\;,, & = & 5+4\times 2 & = & 13. \end{array}$$

That is, any term of the series is found by adding to 5 a number of *twos;* the number being one less than the number denoting the order of the term in the series.

Now, we wish to express the fact just stated in Algebraical language; we must not, therefore, confine ourselves to any particular term, as the 5th, 6th, 10th, but we must adopt such a representation of the order of the term as shall be true for any term that can be taken.

Let, then, $n$ stand for the order of the term.

By what was said above, the $n^{\text{th}}$ term will consist of 5, together with a number of twos, one less than the number $n$; this number, then, must be denoted by $(n - 1)$, and the quantity to be added must be $(n - 1)\,2$; and the expression for the $n^{\text{th}}$ term

$$5 + (n - 1)\,2.$$

And this is the formula for the general term of the particular series

$$5,\ 7,\ 9,\ \&c.$$

This, however, is not a general formula for all series, since 5 and 2 are numbers belonging to the particular series 5, 7, 9.

If, however, we take the series

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \&c.$$

This may serve as the general representative of all series, since $a$ may stand for any 1st term, and $d$ for any common difference.

Now, the $n^{\text{th}}$ term of this $= a + (n - 1)\,d$.

It is usual to denote the $n^{\text{th}}$ term by the letter $l$, we then have

$$l = a + (n - 1)\,d, \qquad (1.)$$

And this formula suffices to find the value of any given term of any given series, by giving to $a$, $n$, $d$, the particular values required.

**145.** We may regard (1) as an equation of one unknown quantity: in every problem to which it applies, three of the quantities $l$, $a$, $n$, $d$ are given; the fourth quantity is then the unknown quantity, and may be found by solving the equation in the usual way.

In fact, there are four varieties of problems which can be solved by this formula.

| | | | | |
|---|---|---|---|---|
| 1°. | When $l$, $a$, $n$ | are given, and | $d$ | is to be found. |
| 2°. | When $l$, $a$, $d$ | „ | $n$ | „ |
| 3°. | When $l$, $n$, $d$ | „ | $a$ | „ |
| 4°. | When $a$, $n$, $d$ | „ | $l$ | „ |

Example of the 1st variety, $l$, $a$, $n$, given.

The first term of a series is 4, the last is 28, and the number of terms is 7: find the common difference.

Generally $\quad l = a + (n - 1)\,d$

In this case $\quad 28 = 4 + 6d$

$\therefore \quad 6d = 24$

$d = 4$, the common difference.

Or, the series is 4, 8, 12, 16, &c.

Example of the 2nd variety, $l$, $a$, $d$, given.

The first and last terms are 40 and 80 respectively, and the common difference is 5: find the number of terms.

$$\begin{array}{lrcl} \text{Generally} & l & = & a+(n-1)\,d \\ \text{In this case} & 80 & = & 40+(n-1)\,5 \\ \therefore & 5n & = & 45 \\ \text{And} & n & = & 9, \text{ the number of terms.} \end{array}$$

Or, the series is 40, 45 . . . . . . 75, 80.

Examples of the 3rd variety, $l$, $n$, $d$, given.

The last term of a series of 6 terms is 59, and the common difference is 3: find the first term.

$$\begin{array}{lrcl} \text{Generally} & l & = & a+(n-1)\,d \\ \text{In this case} & 59 & = & a+5\times 3 \\ \therefore & a & = & 44. \end{array}$$

Or, the series is 44, 47, 50, &c.

Example of the 4th variety, $a$, $n$, $d$, given.

The series $\frac{3x-2a}{2}$, $\frac{5x-3a}{4}$, $\frac{2x-a}{2}$, &c., is continued to 50 terms: find the last term.

To find $d$, we must subtract one of the terms from the preceding, it will be found to be $\frac{a-x}{4}$;

$$\begin{array}{lrcl} \text{Generally} & l & = & a+(n-1)\,d, \\ \text{In this case} & l & = & \dfrac{3x-2a}{2}+49\,\dfrac{a-x}{4} \\ & & = & \dfrac{45a-43x}{4}. \end{array}$$

**146.** We now proceed to find a general expression for the sum of $n$ terms of an Arithmetic progression.

$$a+(a+d)+(a+2d)+(a+3d)+\&c.$$

If we denote the sum by $S$, we have

$$S = a+(a+d)+(a+2d)+\,.\,.+(a+\overline{n-2}d)+(a+\overline{n-1}d).$$

Also $S = (a+\overline{n-1}d)+(a+\overline{n-2}d)+(a+\overline{n-3}d)+\ldots\ldots+(a+d)+a$; if we write the terms in reverse order.

Adding these two equations together, we have, by adding each term to the one vertically over it,

$$2S = (2a+\overline{n-1}d)+(2a+\overline{n-1}d)+(2a+\overline{n-1}d)$$
$$+(2a+\overline{n-1}d)+\&c., \text{ to } n \text{ terms;}$$

And these terms are now all equal, and their sum may be expressed by

$$n(2a+\overline{n-1}d),$$

$$\therefore \quad 2S = (2a+\overline{n-1}d)n;$$

$$\text{Or,} \quad S = (2a+\overline{n-1}d)\frac{n}{2}. \qquad (2.)$$

Which is the general formula for the sum of any number of terms of any given Arithmetic progression. Here we have a general relation between four symbols, $s$, $a$, $n$, $d$; and all the remarks made on formula (1) apply to (2).

One example shall here be given of its application.

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Example on formula (2).

If the sum of a series be 290, the number of terms 10, and the common difference 4: find the first term.

$$\begin{aligned} &\text{Generally} & s &= \{2a+\overline{n-1}d\}\frac{n}{2}, \\ &\text{In this case} & 290 &= \{2a+9\times 4\}\,5, \\ &\therefore & 10a &= 110, \\ &\text{And} & a &= 11. \end{aligned}$$

**147.** A third formula of great use may be derived from the former two.

$$\begin{aligned} &\text{Now} & l &= a+\overline{n-1}d, \quad \text{from (1.)} \\ &\therefore & l+a &= 2a+\overline{n-1}d; \\ &\text{But} & s &= \{2a+\overline{n-1}d\}\frac{n}{2}, \quad \text{from (2.)} \\ &\therefore & s &= (a+l)\frac{n}{2}. \qquad (3.) \end{aligned}$$

This formula gives a relation between the first and last term of a series, and the sum of all the terms.

**148.** Examples worked out.

1. Insert 11 Arithmetic means between 20 and $-4$.

The number of terms when the means are inserted will be 13.

$$\begin{aligned} &\text{And generally} & l &= a+\overline{n-1}d, \quad \text{by (1.)} \\ &\text{In this case} & -4 &= 20+12d, \\ &\therefore & d &= -2; \end{aligned}$$

And the series is $20, 18, 16 \ldots\ldots 2, 0, -2, -4.$

2. If the sum of $n$ terms be $45p$, the first term be $(p-q)$, and the last term $(p+q)$: find the number of terms.

$$\text{Generally} \qquad s = (a+b)\frac{n}{2}, \quad \text{by (3.)}$$

$$\text{In this case,} \qquad 45p = (\overline{p-q} + \overline{p+q})\frac{n}{2};$$

$$\therefore \qquad 45p = np,$$

$$\text{Or,} \qquad n = 45, \quad \text{the number of terms.}$$

*Note.* This result is true whenever we give any numerical values that we please to $p$ and $q$.

3. There is a certain Arithmetic series, such that the sum of any number of its terms taken from the beginning shall be equal to the square of the number of the terms: find the series.

Take the usual notation,

$$\text{Then by (2)} \qquad s = \{2a + \overline{n-1}d\}\frac{n}{2} = n^2 \text{ by the question;}$$

$$\therefore \qquad 2a + \overline{n-1}d = 2n.$$

Since this equation is always true, whatever number $n$ may be,

$$\text{Let} \qquad n = 1,$$

$$\text{Then} \qquad 2a = 2;$$

$$\text{Or,} \qquad a = 1, \quad \text{the first term.}$$

And the above equation becomes

$$(n-1)d = 2(n-1),$$

$$\therefore \qquad d = 2, \quad \text{the common difference.}$$

And the series is $1, 3, 5, 7 \ldots . (2n-1)$.

The problem may be worked similarly, when the sum of the terms is any multiple of $n^2$.

4. The sum of $n$ terms of an A. P. is 120, the first term is 3, and the common difference 2: find the number of terms.

$$\text{Generally} \qquad s = (2a + \overline{n-1}d)\frac{n}{2},$$

$$\text{In this case} \qquad 120 = (6 + 2n - 2)\frac{n}{2};$$

$$\text{Or,} \qquad n^2 + 2n = 120,$$

$$\therefore \qquad n + 1 = \pm 11.$$

$$\left.\begin{array}{ll}\text{And} & n = 10 \\ \text{Or,} & n = -12\end{array}\right\}$$

The first value 10 gives a series of 10 terms

3, 5, 7, 9, 11, 13, 15, 17, 19, 21.

We cannot, however, say that the 2nd value gives a series of $-12$ terms, for that has no meaning. But if we continue the series backwards two places, it becomes

$-1$, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21,

whose sum is the same as the former, and it has 12 terms.

**149.** The reasoning upon which this interpretation is founded is given below.

The formula $$s = (2a + \overline{n - 1}d)\frac{n}{2}$$

may be put in the form

$$n^2 + \frac{2a - d}{d}n = \frac{2s}{d}. \qquad (1.)$$

$a$, $d$, $s$ being known, this is a quadratic in $n$, and its two values are

$$n = -\frac{2a - d}{2d} \pm \frac{\sqrt{4a^2 - 4ad + d^2 + 8sd}}{2d},$$

one of which is generally positive, the other negative.

Now, if we take the equation

$$s = (\overline{2d - a} + \overline{n - 1}d)\frac{n}{2},$$

it may be put in the form

$$n^2 - \frac{2a - d}{2d}n = \frac{2s}{d}. \qquad (2.)$$

and the roots of this are

$$n = \frac{2a - d}{2d} \pm \frac{\sqrt{4a^2 - 4ad + d^2 + 8sd}}{2d},$$

which are the same numerically as the roots of (1), but of contrary signs; that is, the positive value of (2) is the same *numerically* as the negative root of (1); and the second side of (2) is the expression for the sum of a series whose first term is $(d - a)$, and number of terms equal to the positive root of (2), which is the same numerically as the negative root of (1.)

Hence the series answering to the negative value of $n$ is one whose first term is $(d - a)$, common difference $d$, and number of terms the same as that negative value with its sign changed.

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## Examples LXV.

1. Find the last term of the series 3, 5, 7, &c., of 20 terms.

2. Write down the series whose first term is $2a$, common difference $c$; find the expression for the $n^{\text{th}}$ term of this; also the sum of 18 terms.

3. If the 8th and 12th terms of a series be $a$ and $b$ respectively, find the common difference, the first term, and the sum of $a$ terms.

4. Find the Arithmetic mean between $a + b$ and $a - b$, and insert 12 Arithmetic means between $a$ and $b$.

5. If a man begin a journey by walking 30 miles a-day, and fall off 2 miles every day, how long will he be walking 150 miles? Explain the two values.

6. The three sides of a triangle are in A. P., and the greatest side is double of the least: find the ratios of the sides to each other.

7. The three sides of a right angled triangle are in A. P.: find the ratios of the sides to each other.

8. Find the relation between 3 quantities, $a$, $b$, $c$, when they are in A. P. Is one relation sufficient to ensure their being in A.P.? Also, find the conditions that 4 quantities, $a$, $b$, $c$, $d$, may be in A. P.

9. Show that in any Arithmetic progression, whose terms are all of the same sign, no term after the second can be a multiple of the term preceding it.

10. The angles of a triangle are in A. P., and the least is 45°: find the rest. Also, show that if one angle of a four-sided rectilineal figure be a right angle, the angles cannot be in A. P.

11. If there be two or more Arithmetic series, each of the same number of terms; and if the corresponding terms of these series be added together, the resulting terms will form an A. P.

> Show that if the terms of an A. P. be taken at equal intervals, they will form an A. P.

12. In an A. P., the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.

13. Sum the series $(b + c)^2 + (b^2 + c^2) + (b - c)^2 +$ &c., to $n$ terms.

14. The series 1, 3, 5, 7, &c., and the series 2, 4, 6, 8, &c., are each continued to the same number of terms; and the ratio of the former sum to the latter sum is 10 : 11; find to how many terms each is continued. Also, find the ratio which the sum of $n$ terms of the first series bears to $n$ terms of the second series.

15. If, in an A. P., the sum of $n$ terms always equals $2n^2$: find the series. Also, when the sum of $n$ terms $= pn^2$.

## GEOMETRIC PROGRESSION.

**150.** *Def.*—A "*Geometric Progression*" is a series of terms more than two, which go on increasing or decreasing in such a manner that the ratio of any term to that preceding it is always the same, from whatever part of the series the two terms be taken.

Thus 4, 12, 36, &c., is a G. P.

Also $-5, 10, -20, 40,$ &c. „

$\frac{3a}{2x}, \sqrt{\frac{3}{2}}, \frac{x}{a},$ &c. „

**151.** *Def.*—The ratio of any term to the preceding is called the "*Common Ratio.*"

Thus 3, $-2$, $\frac{x}{a}\sqrt{\frac{2}{3}}$, are the common ratios in each of the three above series respectively.

**152.** To express the general term of a G. P. in terms of the 1st term and the common ratio.

Let the given series be

$$a, ar, ar^2 \dots \&c.$$

Let $l$ denote the $n^{th}$ or general term, and $n$ the order of the term; $r$ the common ratio.

It is clear, from the construction of the series, that the index of the $n^{th}$ term will be $(n-1)$.

$$\therefore \quad l = ar^{n-1} \quad (1)$$

is the general formula of the $n$th term.

If any three of the quantities $l$, $a$, $r$, $n$, be given, the other may be found in the same way as was explained in Arithmetic Progression.

**153.** To find a general expression for the sum of $n$ terms of a Geometric Progression.

Let the series be $a, ar, ar^2 \dots \&c.$

Let $S$ denote the sum of $n$ terms.

$$\text{Then } S = a + ar + ar^2 + \dots ar^{n-3} + ar^{n-2} + ar^{n-1}.$$

$$\therefore Sr = ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} + ar^n.$$

By subtracting the upper from the lower we get

$$S(r-1) = ar^n - a,$$

$$\text{Or,} \quad S(r-1) = a(r^n - 1);$$

$$\therefore \quad S = a\cdot\frac{r^n-1}{r-1}. \quad (2.)$$

If numerator and denominator of the second side of (2) be multiplied by $(-1)$, we get

$$S = a\cdot\frac{1-r^n}{1-r}. \quad (3.)$$

If $r$ be greater than 1, formula (2) should be used.

If $r$ be less than 1, „ (3) „

**154.** Examples of the use of each formula will now be given.

Ex. 1. The first term of a geometric series is 3, the number of terms 10, and the 10th term is 1536; find the common ratio.

Generally $l = a.r^{n-1}$, by (1.)

In this case $1536 = 3.r^9$.

$\therefore \quad r^9 = 512$;

Or, $\quad r^9 = 2^9$.

Taking the 9th root of both sides, we find $\Big\} \; r = 2$, the common ratio.

And the series is 3, 6, 12, 24 . . . . . . . 768, 1536.

Ex. 2. When the 1st term of a geometric series is 27, the common ratio $\frac{1}{3}$, and the last term $\frac{1}{81}$, find the number of terms.

Generally $\quad l = ar^{n-1}$.

In this case $\quad \frac{1}{81} = 27\left(\frac{1}{3}\right)^{n-1}$,

$\therefore \quad \frac{1}{3^{n-1}} = \frac{1}{81 \times 27}$;

Or, $\quad \frac{1}{3^{n-1}} = \frac{1}{3^7}$.

$\therefore \quad n - 1 = 7$;

Or, $\quad n = 8$, the number of terms.

The series therefore is $27, 9 \; . \; . \; . \; . \; \frac{1}{27}, \frac{1}{81}$.

The two other cases of formula (1) are when *lrn* or *arn* are given.

The next two examples are on formula (2).

Ex. 3. The common ratio is 4, and the sum of 5 terms is 682; find the number of terms.

Generally $\quad S = a.\frac{r^n - 1}{r - 1}$.

In this case $\quad 682 = a.\frac{4^5 - 1}{4 - 1} = a.\left(\frac{1024 - 1}{4 - 1}\right)$;

Or, $\quad a = \frac{682}{341}$.

$\therefore \quad a = 2$, the first term.

And the series is 2, 8, 32, 128, 512.

Ex. 4. Find the sum of 10 terms of this series

$$x, \; \sqrt{\frac{x}{a}}, \; \frac{1}{a}, \; \text{\&c.}$$

Here we must first find $r$.

Now $\quad \sqrt{\frac{x}{a}} \div x = r$;

Or, $$\frac{1}{x} \cdot \frac{\sqrt{x}}{\sqrt{a}} = r;$$

$$\therefore \quad r = \frac{1}{\sqrt{ax}}.$$

Then we have given $a$, $n$, $r$.

Generally $$s = a \cdot \frac{r^n - 1}{r - 1}.$$

In this case $$s = x \cdot \frac{\left(\frac{1}{\sqrt{ax}}\right)^{10} - 1}{\frac{1}{\sqrt{ax}} - 1}.$$

Or, $$s = \frac{x}{(ax)^{\frac{9}{2}}} \cdot \frac{1 - (ax)^{\frac{10}{2}}}{1 - (ax)^{\frac{1}{2}}},$$

$$s = \frac{1 - (ax)^5}{(a^9x^7)^{\frac{1}{2}}\{1 - (ax)^{\frac{1}{2}}\}}.$$

Ex. 5. If a population of 50,000 be increased in 2 years to 53,045, find what the population will be at the end of 5 years.

Let $r$ = the ratio which the population at the beginning of any one year bears to that at the beginning of the preceding year,

Then $$50{,}000r^2 = 53{,}045;$$

$$\therefore \quad r^2 = \frac{53{,}045}{50{,}000}$$

And $$r = \frac{103}{100};$$

$$\therefore \quad 50{,}000 \times \left(\frac{103}{100}\right)^5 = \text{the population at the end of 5 years.}$$
$$= 57{,}964 \text{ nearly.}$$

Ex. 6. If $S_n$ denote the sum of $n$ terms of a G. P., whose common ratio is 2, and if $S_{2n} \div S^n = 129$ find $n$.

Here $$\frac{S_{2n}}{S_n} = \frac{a(r^{2n} - 1)}{r - 1} \div a \cdot \frac{r^n - 1}{r - 1};$$

$$\therefore \quad 129 = (r^n + 1),$$

$$\therefore \quad r^n = 128;$$

Or, $$2^n = 128.$$

$$\therefore \quad n = 7.$$

## Examples LXVI.

1. Find the 20th term of the series 2, 4, 8, &c.; the 16th term of 5, 20, 80, &c.; the $n^{\text{th}}$ term of $3x$, $9x^2$, $27x^3$, &c.; the $n^{\text{th}}$ term of $2 - x + \frac{x^2}{2}$, &c.; the $n^{\text{th}}$ term of $(a - x)$, $(a - x)^3$, $(a - x)^5$, &c.; the $n^{\text{th}}$ term of $\frac{a}{x^2}$, $\frac{1}{x}$, $\frac{1}{a}$, &c.; the $(3x + 1)$th term of $- 6 + 3 - \frac{3}{2} +$, &c.

2. Sum the following Geometric series, $7 + 21 + 63 +$, &c., to 19 terms; $5 + 10 + 20 +$, &c., to 10 terms; $\frac{1}{2} + \frac{3}{2} + \frac{9}{2}$, &c., to 8 terms; $4 + 2 + 1$, &c., to 16 terms and to $n$ terms; $6 + 4 + 2\frac{2}{3}$, &c., to 30 terms and to $3n$ terms; $\frac{x - 1}{x} + 1 + \frac{x}{x - 1} +$, &c., $(a^2 + ax + x^2) + (a^3 - x^3) + (a^4 - a^3x - ax^3 + x^4)$, &c., to $(n + 1)$ terms; $(a + x)^2 + (a^2 - x^2) + (a - x)^2 +$, &c., to $2n$ terms; $(x - 1)^2 - (x^2 - 1) + (x + 1)^2$ to $n$ terms; $(\sqrt{2} - 1) + 1 + (\sqrt{2} + 1)$, &c., to $n$ terms.

3. Insert 5 Geometric means between 2 and 128; also 8 G. M. between $\frac{1}{3}$ and 9; 16 G. M. between $- 4$ and $- \frac{1}{32768}$; 23 G. M. between $(a + b)^{12}$ and $(a + b)^{-12}$; 16 G. M. between $a^2x$ and $\frac{1}{a^3x^3}$.

4. If ten terms be in G. P., and the last be 1024 times the first, find the common ratio. Also find the relation between the 2nd and the 8th term.

5. If the 20th term of a series be 64, and the 24th term be $\frac{625}{4}$, find the 1st term and the 100th term.

6. The $m^{\text{th}}$ term of a Geometric series is $a$, and the $n^{\text{th}}$ term is $b$; find the first term.

7. The sum of the first three terms of a Geometric series is double of the first term: find the common ratio.

8. If the sum of $x$ terms of a Geometric series be 4 times the sum of $\left(\frac{x}{2}\right)$ terms, find the value of $r$. Also if the sum of five terms of a Geometric series be 33 times the sum of five terms, find the value of $r$. Is the first term of the series determined in either of these cases?

9. The population of Great Britain in 1841 was 27,019,548; in 1851 it was 27,770,091. The ratio of increase being supposed uniform, find the increase each year in the population; also the rate of increase per cent. per annum.

10. If a sum of money be left to accumulate at compound interest, the principals of the successive years form a Geometric series. State what is the first term and what the ratio of each principal to the preceding one; also if 16,000*l.* be put to interest for 4 years, and amount

to 19,448*l*. 2*s*. at compound interest; find the rate per cent. per annum of interest.

11. If the areas of a series of squares be in G. P., prove that their sides will also be in G. P.

12. If $s_n$ denote the sum of a series of $n$ terms of a G. P., show that $s_n + rs_{n-2} = (1 + r)s_{n-2}$; also show that $s_{n+1} - s = rs_n$.

13. Find the sum of the following series to $n$ terms; $1 . 2 + 3 . 6 + 9 . 18 +$, &c.; $\frac{1}{2 . 3^2} + \frac{1}{2^2 . 3} + \frac{1}{2^3 . 1} +$, &c.

14. Show that the sum of the terms of a G. P. is less than $(a + b)^{\frac{n}{2}}$, also that the product of $n$ terms of an A. P. is greater than $(ab)^{\frac{n}{2}}$, $a$, $b$, denoting the 1st and $n$th terms respectively.

15. Find a number such that the Arithmetic mean between it and 20 shall be three times the Geometric mean.

## INFINITE GEOMETRIC SERIES.

**155.** If the value of $r$ in a Geometric series be greater than unity, the terms continually increase, and if the number of terms be great enough, the last term may be made greater than any assigned number; and of course the sum of $n$ terms will also be greater than any assigned number. It is therefore said that when $r > 1$ and $n$ increases without limit, the $n$th term and the sum of $n$ terms increase without limit.

Ex. Take the series 3, 6, 12, &c.

If a number, say ten million, be proposed, it is possible to take so many terms, that the last of them shall be greater than ten million, and similarly if any greater number be proposed.

If $r$ be equal to unity, it is evident that all the terms are alike, and the last term is not greater or less than any other term; but in this case the sum of $n$ terms can be made as great as we please, by giving a sufficiently large value to $n$.

If $r$ be less than unity, the terms continually decrease, and if the number of terms be sufficiently great, the last or $n$th term may be made less than any assigned number. This may be seen from the general *expression* from the $n$th term, which is $ar^{n-1}$.

Now $r$ can always be put in the form $\frac{1}{1+k}$, when $r < 1$.

And $$ar^{n-1} = \frac{a}{(1+k)^{n-1}}.$$

And it is evident that $(1+k)^{n-1}$ can be made as large as we please by taking $n$ large enough, and therefore $\frac{a}{(1+k)^{n-1}}$ can be made less than any assigned number.

Ex. Take the series 3, 2, $\frac{4}{3}$, &c.; and let a small fraction, say .000000001, be proposed, it is possible to find a term of the series which shall be less than this.

To find the sum of $n$ terms in this case we have the formula

$$s = a \cdot \frac{1 - r^n}{1 - r}.$$

We now proceed to discuss the value of $s$ when $n$ is made indefinitely great.

The above formula may be written thus,

$$s = \frac{a}{1-r} - \left(\frac{a}{1-r}\right) \cdot r^n.$$

Now the latter part of this expression $\left(\frac{a}{1-r}\right) \cdot r^n$ becomes indefinitely small when $n$ is made indefinitely great, or the difference between the true value of $s$ and the quantity $\frac{a}{1-r}$ may be made less than any assignable number, by sufficiently increasing $n$. Hence $\frac{a}{1-r}$ is called the "*Limit*" to which the sum of the series approaches as $n$ is indefinitely increased.

Thus the limit of the sum of the series

$$2 + \tfrac{1}{2} + \tfrac{1}{8} +, \text{\&c.} \ldots \textit{ad infinitum}$$

is $$2 \cdot \frac{1}{1 - \frac{1}{4}};$$

or, $$\tfrac{8}{3}.$$

That is, $\frac{8}{3}$ is the limit to which the sum of the series may be made to approach as nearly as we please by taking a sufficient number of terms.

The difference between the "Limit" of the series and the actual sum of any number of terms ($n$) is always represented by the quantity

$$\left(\frac{a}{1-r}\right) \cdot r^n,$$

which in the case just given becomes

$$\frac{8}{3}\left(\frac{1}{4}\right)^n = \frac{1}{3 \,.\, 2^{2n-3}}.$$

The student must carefully remember that when any *finite* number of terms of the series

$a,\ ar,\ ar^2$, &c., are taken,

the value always is $\dfrac{a}{1-r} - \dfrac{a}{1-r}\,r^n$

and the expression $\dfrac{a}{1-r}$ is never the exact value of the sum of any number of terms. All that must be understood is this: $\dfrac{a}{1-r}$ is the limit to which the actual sum of the series may be made to approximate as nearly as we please; and no other quantity but itself has this property.

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## Examples LXVII.

1. Find the limit of the following series when continued *ad infinitum*:—

$1 + \frac{1}{2} + \frac{1}{4}$, &c.; $3 + 1 + \frac{1}{3} +$, &c.; $\frac{1}{9} + \frac{1}{81} + \frac{1}{729} +$, &c.;

$6 + 3 + \frac{3}{2}$, &c.; $4 + 3 + 2\frac{1}{4}$, &c.; $ar + a^2r^2 + a^3r^3$, &c., when $ar < 1$.

2. Find the limit of

$1 - \frac{1}{2} + \frac{1}{4}$, &c., ad inf.; of $1 - \frac{1}{4} + \frac{1}{16} -$, &c., *ad inf.*

Of $\frac{1}{8} + \frac{1}{16} + \frac{1}{32}$, &c., to $n$ terms, and also *ad inf.*

3. Find the difference between 200 terms and the limit of the series $24 + 9 + \frac{27}{8}$, &c.

4. Find the value of the Recurring Decimals, $.\dot{3}$, $.\dot{4}\dot{5}$, $.00\dot{6}$, $.\dot{1}2\dot{3}$, $.\dot{4}2857\dot{1}$, $.00\dot{6}\dot{3}$. How far will the quantity $.00636363$ differ from the last limit?

5. How many places of decimals must be taken in order that $.\dot{7}\dot{5}$ may differ from it by less than one hundred-millionth part of unity.

6. Find the value of $n$ terms and *ad inf.* of the series

$$\frac{1}{2^3 \,.\, 1} + \frac{1}{2^2 \,.\, 3} + \frac{1}{2 \,.\, 3^3} +, \text{ \&c.}; \quad \frac{1}{1 \,.\, 2 \,.\, 3} + \frac{1}{2 \,.\, 4 \,.\, 6} + \frac{1}{4 \,.\, 8 \,.\, 12} +, \text{ \&c.}$$

$$\frac{1}{3 \,.\, 5} - \frac{1}{6 \,.\, 10} + \frac{1}{12 \,.\, 20} -, \text{ \&c.}$$

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## HARMONIC PROGRESSION.

**156.** *Def.*—A Harmonic Progression is a series of terms more than two, and such, that if any three consecutive terms be taken, the difference between the first and second of these is to the difference between the second and third as the first is to the third.

Thus, if $A$, $B$, $C$, be in Harmonic Progression,

Then $$A - B : B - C :: A : C$$

Also the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{.}$, $\frac{1}{a+b}$, $\frac{1}{a}$, $\frac{1}{a-b}$, $\frac{1}{a-2b}$, &c.; 9, $4\frac{1}{2}$, 3, $2\frac{1}{4}$, &c., are examples of Harmonic series.

**157.** A property of Harmonic series will now be proved, which furnishes an easy test to discover whether a proposed series be Harmonic or not.

*If a series of quantities be in* H. P., *their reciprocals are in* A. P.

To prove this generally, suppose $a$, $b$, $c$ to be three of the terms of the Harmonic series.

Then $a - b : b - c :: a : c$, by Def.

$$\therefore \qquad \frac{a-b}{a} = \frac{b-c}{c}.$$

And $$\frac{a-b}{ab} = \frac{b-c}{bc},$$

$$\therefore \qquad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Or, the difference between the reciprocal of the first and of the second is equal to the difference between the reciprocal of the second and of the third.

Therefore, the reciprocals of $a$, $b$, $c$, form an A. P.

Problems in H. P. will be most easily solved by making use of this proposition.

**158.** To find the $n^{\text{th}}$ term of a given Harmonic series.

Let the series be $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$, &c.

It is evident that the $n^{\text{th}}$ term will be the reciprocal of the $n^{\text{th}}$ term of the Arithmetic series

$$a, \ \overline{a+b}, \ \overline{a+2b}, \ \&c.,$$

Or, the $n^{\text{th}}$ term $= \dfrac{1}{x+\overline{n-1}\,b}$.

It may be remarked that every Harmonic series may be put in the above form. For example, the series 6, 4, 3, may be written

$$\frac{1}{(\frac{1}{6})},\ \frac{1}{(\frac{1}{4})},\ \frac{1}{(\frac{1}{3})}.$$

If it be required to insert any number ($n$) of Harmonic means between two quantities $a$ and $b$, we must proceed as follows :

Let $a$ and $b$ be the two extremes.

Then we must first insert $n$ Arithmetic means between $\dfrac{1}{a}$ and $\dfrac{1}{b}$; and the reciprocals of these will be the Harmonic means required.

The Harmonic series in this case will be

$$a,\ \frac{(n+1)\,ab}{b\,(n+2)-a},\ \frac{(n+1)\,ab}{b\,(n+3)-a},\ \text{\&c.}$$

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### Examples LXVIII.

1. If $a, b, c$, be in H. P. show that $a^2+c^2 > 2b^2$; also, show that $a^n + c^n > 2b^n$, where $n$ is any positive integer.

2. From the equations

$$(x-a)(x-b) = (a-c)(x-d) = (x-e)^2$$

find the values of $e$. Also, if $e'$, $e''$ be the two values of $e$, show that $e''-e'$ is a Harmonic mean between $a-e'$ and $b-e'$.

3. If $a, b, c$ be in H. P., show that

$$\frac{1}{a}+\frac{1}{c} = \frac{1}{b-a}+\frac{1}{b-c}.$$

4. The sum of two numbers, together with their Arithmetic mean and Geometric mean is 15, and the product of the four quantities is 144 : find the numbers.

5. If $b+c$, $c+2$, $a+b$, be in H. P., then $a^2$, $b^2$, $c^2$, shall be in A. P.

6. The difference between the Arithmetic and Geometric means of two numbers is $a$, and that between the Arithmetic and Harmonic means is $b$ : find the numbers.

**159.** Some examples will now be given of the Summation of Series, which are neither purely Arithmetical nor purely Geometrical, but which may be resolved into the sum or difference, or some other function of those.

Ex. 1. Sum the series $11 + 15 + 23 + 39 +$, &c. to $n$ terms.

$$\begin{aligned} \text{Sum} &= (4+7) + (8+7) + 16 + 7) + \ldots\ldots + 2^{n+1} + 7 \\ &= (4 + 8 + 16 + \ldots\ldots + 2^{n+1}) + 7n \\ &= 4 \cdot \frac{2^n - 1}{2 - 1} + 7n; \\ \therefore\ S &= 2^{n+2} + 7n - 4. \end{aligned}$$

Ex. 2. Sum the series $8 + 16 + 36 + 92 + 256 +$, &c. to $n$ terms.

$$\begin{aligned} \text{Here,}\quad \text{sum} &= (3 + 9 + 27 + 81 + 243 +, \text{\&c.}) \\ &\quad + 5 + 7 + 9 + 11 + 13 +, \text{\&c.}, \end{aligned}$$

which is the sum of a Geometric and an Arithmetic series.

Ex. 3. Sum the series $a^2 + (a + b)^2 + (a + 2b)^2 +$, &c., to $n + 1$ terms.

$$\begin{aligned} \text{Here}\quad (a+b)^3 - a^3 &= 3a^2b + 3ab^2 + b^3 \\ (a+2b)^3 - (a+b)^3 &= 3(a+b)^2b + 3(a+b)b^2 + b^3 \\ (a+3b)^3 - (a+2b)^3 &= 3(a+2b)^2b + 3(a+2b)b^2 + b^3 \\ \text{\&c.}\qquad &\qquad \text{\&c.}\qquad\qquad \text{\&c.} \end{aligned}$$

$$(a + nb)^3 + (a + \overline{n-1}b)^3 = 3(a + \overline{n-1}b)^2b + 3(a + \overline{n-1}b)b^2 + b^3,$$

$$(a + \overline{n+1}b)^3 - (a + nb)^3 = 3(a + nb)^2b + 3(a + nb)b^2 + b^3.$$

Adding all these equations together, we get

$$\begin{aligned} (a + \overline{n+1}b)^3 - a^3 &= 3b\{a^2 + (a+b)^2 + \ldots\ldots + (a+nb)^2\} \\ &\quad + 3b^2\{a + (a+b) + \ldots\ldots + (a+nb)\} \\ &\quad + (n + 1b^3. \end{aligned}$$

Hence we find $a^2 + (a+b)^2 + \ldots\ldots + (a + \overline{n-1}b)^2 + (a+nb)^2$

$$= \frac{1}{3b}\left\{(a + \overline{n+1}b)^3 - a^3 - 3b^2(2a + nb)\frac{n+1}{2} - nb^3\right\}$$

$$\text{Or,}\quad S = \frac{n+1}{6}\left\{6a^2 + 6nab + n(2n+1)b^2\right\}.$$

By a similar process, we may find the sum of such series as

$$1^2 + 2^2 + 3^2 +, \text{\&c.}$$
$$5^2 + 8^2 + 11^2 +, \text{\&c.}$$

For these are, in fact, only particular cases of the general formula above.

Ex. 4. Sum the series $1.6 + 20.25 + 39.44 +$, &c., to $n$ terms.

The series may be put in the form

$$\begin{aligned} &(6 - 5)6 + (25 - 5)25 + (44 - 5)44 +, \text{\&c.} \\ &= 6^2 + 25^2 + 44^2 +, \text{\&c.} \\ &- 5(6 + 25 + 44 +, \text{\&c.}); \end{aligned}$$

The first of which series may be summed by the method of Ex. 3, and the second is an Arithmetical series.

Ex. 5. To sum a series consisting of the cubes of numbers in Arithmetical progression, as $a^3 + (a+b)^3 + (a+2b)^3 +$, &c. to $n$ terms.

We have $(a+b)^4 - a^4 = 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+2b)^3 - (a+b)^3 = 4(a+b)^3 b - 6(a+b)^2 b^2 + 4(a+b) b^3 + b^4.$ &c. &c.

And by carrying on these equations, as in Ex. 3, and by using the result there obtained for the sum of the squares of $a$, $a+b$, &c., we shall arrive at an expression for the series of cubes.

The same method may be used for summing a series consisting of higher powers (as the 4th, 5th, &c.) of numbers in Arithmetic progression, but it then becomes very laborious.

Ex. 6. To sum the series

$$2.7.15 + 4.9.17 + 6.11.19 +, \text{\&c.}$$

This may be put in the form

$$(7-5)7(7+8) + (9-5)9(9+8) + (11-5)11(11+8) +, \text{\&c.}$$
$$= (7^3 + 3.7^2 - 40.7) + (9^3 + 3.9^2 - 40.9) +, \text{\&c.}$$
$$= (7^3 + 9^3 + 11^3 +, \text{\&c.}) + 3(7^2 + 9^2 + 11^2 +, \text{\&c.}) - 40(7 + 9 + 11 +, \text{\&c.})$$

And each of these separate series may be summed by one of the preceding formulæ, and the whole sum becomes known.

Ex. 7. Sum the series

$$\frac{1}{(a+\sqrt{b})(2a+\sqrt{b})} + \frac{1}{(2a+\sqrt{b})(3a+\sqrt{b})} +, \text{\&c.}$$

Now $\dfrac{1}{a+\sqrt{b}} = \dfrac{2a+\sqrt{b}}{(a+\sqrt{b})(2a+\sqrt{b})} = \dfrac{1}{2a+\sqrt{b}} + \dfrac{1}{(a+\sqrt{b})(2a+\sqrt{b})}$.

So $\dfrac{1}{2a+\sqrt{b}} = \dfrac{1}{3a+\sqrt{b}} + \dfrac{1}{(2a+\sqrt{b})(3a+\sqrt{b})}$

$$\frac{1}{na+\sqrt{b}} = \frac{1}{(n+1)a+\sqrt{b}} + \frac{1}{(na+\sqrt{b})(1n+1a+\sqrt{b})}$$

∴ By Addition,

$$\frac{1}{a+\sqrt{b}} - \frac{1}{(n+1)a+\sqrt{b}} = \text{the sum required.}$$

Ex. 8. Sum the series $\dfrac{1}{5.6} + \dfrac{1}{6.7} + \dfrac{1}{7.8} +$, &c., to $n$ terms.

Now $\dfrac{1}{5} - \dfrac{1}{6} = \dfrac{1}{5.6}$

$$\frac{1}{6} - \frac{1}{7} = \frac{1}{6.7},$$

$$\text{\&c.} \qquad \text{\&c.}$$

$$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{(n-1)n}$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Adding all these equations together, we get

$$\frac{1}{5} - \frac{1}{n+1} = \frac{1}{5.6} + \frac{1}{6.7} + \ldots\ldots + \frac{1}{n(n+1)};$$

Or, $$S = \frac{n-4}{5(n+1)}.$$

Ex. 9. Sum the series

$$\frac{1}{3.7.11} + \frac{1}{7.11.15} + \frac{1}{11.15.19} + \ldots + \frac{1}{(4n-1)(4n+3)(4n+7)}.$$

Now $$\frac{1}{3.7} - \frac{1}{7.11} = \frac{8}{3.7.11}$$

$$\frac{1}{7.11} - \frac{1}{11.15} = \frac{8}{7.11.15}$$

$$\frac{1}{11.15} - \frac{1}{15.19} = \frac{8}{11.15.19}$$

$$\text{\&c.} \qquad \text{\&c.}$$

$$\frac{1}{(4n-1)(4n+3)} - \frac{1}{(4n+3)(4n+7)} = \frac{8}{(4n-1)(4n+3)(4n+7)}.$$

By Addition,

$$\frac{1}{3.7} - \frac{1}{(4n+3)(4n+7)} = 8\left\{\frac{1}{3.7.11} + \frac{1}{7.11.15} + \text{, \&c.}\right\}$$

$$\therefore \qquad S = \frac{1}{8}\left\{\frac{1}{3.7} - \frac{1}{(4n+3)(4n+7)}\right\}.$$

Some series which cannot be treated by any of the above methods will be discussed in Chapter (xvii.).

Ex. 10. Sum the series $1 + 5 + 12 + 22 + 35 + 51 +$, &c., to $n$ terms.

Let $A_1, A_2, A_3 \ldots A_n$ denote the 1st, 2nd, 3rd, ... $n^{\text{th}}$ terms,

Then $$A_n = A_{n-1} + 3n - 2$$

$$A_{n-1} = A_{n-2} + 3(n-1) - 2,$$

$$\text{\&c.} \qquad \text{\&c.}$$

$$A_3 = A_2 + 3.3 - 2$$

$$A_2 = A_1 + 3.2 - 2.$$

By Addition, $A_n = A_1 + 3(n + \overline{n-1} + \ldots + 2) - 2(n+1)$,

Or, $A_n = 1 + 3 . \frac{n^2 + n - 2}{2} - 2n + 2$,

$$\therefore \quad A_n = \frac{3n^2}{2} - \frac{n}{2}.$$

Similarly, $A_{n-1} = \frac{3}{2}(n-1)^2 - \frac{n-1}{2}$,

&c. &c.

$$A_3 = \frac{3}{2}(3)^2 - \frac{3}{2}$$

$$A_2 = \frac{3}{2} . (2)^2 - \frac{2}{2}$$

$$A_1 = \frac{3}{2}(1)^2 - \frac{1}{2}.$$

By Addition,

$$A_1 + A_2 + \ldots + A_n = \tfrac{3}{2}\{1^2 + 2^2 + 3^2 + \ldots + n^2\}$$
$$- \tfrac{1}{2}\{1 + 2 + 3 +, \&c. + n\},$$

Or, $S = \frac{n^2(n+1)}{2}$.

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### Examples LXIX.

1. Sum the series—

$5 + 9 + 17 + 33 +$, &c., to $n$ terms.
$3 - 2 + 4 - 3 +$, &c., to $2n$ terms.
$5 + 10 + 19 + 36 + 69 +$, &c., to $n$ terms.
$\frac{3}{4} + \frac{7}{8} + \frac{7}{16} +$, &c. $\ldots + \frac{1}{4}\left(\frac{1}{2^{n-1}} + 1 + n\right)$.
$2 + 1 + \frac{1}{2} + \frac{3}{4} + 2\frac{1}{8} +$, &c., to 19 terms.
$- a - a + 3a + 19a + \ldots + a(3^{n-1} - 2n)$.

2. Sum the following series :—

$1^2 + 2^2 + 3^2 + \ldots + n^2$;
$3^2 + 5^2 + 7^2 + \ldots + (2n-3)^2$;
$5^2 + 9^2 + 13^2 +$, &c., to 12 terms;
$1.2 + 2.3 + 3.4 +$, &c., to 19 terms;
$5.12 + 12.19 + 19.26 +$, &c., to $3n$ terms;
$a(a-b) + (a-b)(a-2b) +$, &c., to $n$ terms;
$(x-7y)(x-6y) + (x-6y)(x-5y) +$, &c., to $r$ terms.

3. Sum the series $4 + 16 + 40 + 88 +$, &c., to $r$ terms;

also $a + (a + b) + (a + b + br) + (a + b + br + br^2) +$, &c., to $n$ terms, and *ad inf.* when possible.

$1^3 + 2^3 + 3^3 +$, &c. &c. $+ n^3$;

$a^3 + (a + b)^3 + (a + 2b)^3 +$ to $n$ terms; $2.4.6 + 3.5.7 +$, &c. to $n$ terms; $5.9.11 + 8.12.14 +$, &c., to $n$ terms.

4. Find the sum of each of the following to $n$ terms :—

$$\frac{1}{(a+1)(2a+1)} + \frac{1}{(2a+1)(3a+1)} +, \&c.; \quad \frac{1}{5\,.\,6} + \frac{1}{6\,.\,7} + \frac{1}{7\,.\,8} +, \&c.;$$

$$\frac{1}{6.8} + \frac{1}{7.9} + \frac{1}{8.10} +, \&c.; \quad \frac{1}{5.14} + \frac{1}{8.17} + \frac{1}{11.20} +, \&c.;$$

$$\frac{1}{5.6.7} + \frac{1}{6.7.8} + \frac{1}{7.8.9} +, \&c.; \quad \frac{1}{5.11.17} + \frac{1}{7.13.19} + \frac{1}{9.15\ 21} +, \&c.;$$

$$\frac{1}{x(x+y)(x+2y} + \frac{1}{(x+y)(x+2y)(x+3y)} +, \&c.$$

5. Sum the series whose $n^{\text{th}}$ term is $\dfrac{1}{n(n+4)}$ to $n$ terms, and *ad infinitum.*

6. Also the series whose $n^{\text{th}}$ term $= (n - 3)(n - 2)$.

7. Find the $n^{\text{th}}$ term, and the sum of $n$ terms of a series in which each term is the square of the preceding one, the first term being given.

8. Find the sum of the following, each to $n$ terms :—

$$2.3^2.7 + 3.4^2.8 + 4.5^2.9 +, \&c.;$$
$$4.7^2.9 + 9.12^2.14 + 14.17^2.19 +, \&c.;$$
$$2^3 - 3^3 + 4^3 - 5^3 + 6^3 - 7^3 +, \&c.$$

What does the last series become when $n$ is indefinitely great?

$$1.3^2 + 2.4^2 + 3.5^2 +, \&c.$$

9. Find the sum of 200 terms of the series

$$6 + 13 + 22 + 33 + 46 +, \&c.$$

Also of $n$ terms of the two following :—

$5 + 6 + 8 + 11 + 15 +$, &c., and $7 + 11 + 18 + 28 + 41 +$, &c.

10. Find the 500th term of the series $9 + 28 + 53 + 84 +$, &c., and the $n^{\text{th}}$ term of $2 - 11 - 23 - 34$, &c.

11. Which term of the series $7 + 12 + 19 + 28 +$, &c., amounts to 206?

12. Find the sum of $2n$ terms of the series whose $r^{\text{th}}$ term is $3r^2 - 2r - 8$.

13. Find the sum of $3n$ terms of a series whose first term is 1, and of which each term exceeds the preceding by $2n^2$ ($n$ being the order of the term).

14. Find the sum of the infinite series $a + ar + (a + ab)r^2 + (a + ab + ab^2)r^3 +$, &c.; $r$ being less than 1, and $b$ less than $r^{-1}$.

15. Pearls in Ceylon are sorted into 10 classes, by being riddled through perforated trays, the number of perforations in each tray being 19, 37, 61, 21, &c. Find the $n^{th}$ term of this series, and the sum of $n$ terms.

16. If the series of odd numbers be arranged as follows,

$$\begin{array}{cccc} 1 \\ 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 & 19 \\ & \&c. & \&c. \end{array}$$

prove that the sum of the numbers in the $n^{th}$ line is $n^3$, and that the sum of the first $n$ lines is $\left\{\frac{n(n+1)}{1.2}\right\}^2$.

# CHAPTER XV.

## PERMUTATIONS AND COMBINATIONS.

**161.** Def.—*The "Permutations" of a given number of things are the different arrangements which can be made of them in respect of their order.*

Thus *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, are the permutations of the three letters *abc* taken all together.

Again, *ab*, *ba*, *ac*, *bc*, *ca*, *cb*, are the permutations of *abc* taken two together.

**162.** Def.—*The "Combinations" of a given number of things are the different collections which can be made of them without regard to the order in which they are placed.*

Thus *ab*, *ac*, *bc*, are all the combinations of *abc* taken two at a time.

To find the number of permutations of $n$ things taken two at a time.

Let the $n$ things be the letters *abcd*, &c.; $n$ in number.

Now it is evident that $a$ may be placed before each of the other letters, making the group

$$ab,\ ac,\ ad,\ \&c. \ldots \text{ to } \overline{n-1} \text{ terms.}$$

Also, $b$ may be placed before each of the others, making the group

$$ba,\ bc,\ bd,\ \&c. \ldots \text{ to } \overline{n-1} \text{ terms.}$$

Similarly, all the letters will stand first in their turn, and each group will consist of $\overline{n-1}$ permutations, and the number of groups will be $n$.

Hence the whole number of permutations of $n$ things will be

$$n(n-1);$$

Or, $${}^nP_2 = n(n-1). \qquad (1.)$$

To find the number of permutations of $n$ things, taken three at a time.

Take the $\overline{n-1}$ letters $b,\ c,\ d,$ &c.

The number of permutations of these, taken two together, will be $(n-1)(n-2)$ by formula (1), and they are represented in the table below.

$$\left.\begin{array}{ll} bc,\ bd,\ be,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ cb,\ cd,\ ce,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ db,\ dc,\ de,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ \quad \&c. & \quad \&c. \end{array}\right\} \begin{array}{c} \text{The number of vertical} \\ \text{columns is } \overline{n-1}. \end{array}$$

Let $a$ be placed before each of these, we then have

$$\left.\begin{array}{ll} abc,\ abd,\ abe,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ acb,\ acd,\ ace,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ adb,\ adc,\ ade,\ \&c. \ldots & \text{to } \overline{n-2} \text{ terms} \\ \quad \&c. & \quad \&c. \end{array}\right\}$$

There are thus $(n-1)(n-2)$ permutations of three letters together, in which $a$ stands first.

Similarly, it may be shown that there are $(n-1)(n-2)$ permutations, in which $b$ stands first, and so on for each of the $n$ letters $a,\ b,\ c,\ d,$ &c.

Therefore, the whole number of permutations will be

$$n(n-1)(n-2)$$

Or, $${}^nP_3 = n(n-1)(n-2). \qquad (2.)$$

We shall now show generally that the number of permutations of $n$ things taken $r$ together is

$$n(n-1)(n-2) \ldots . (n-r+1).$$

Let us assume that the number of permutations of the $n$ letters $a,\ b,\ c,\ d,$ &c., taken $\overline{r-1}$ together, is

$$n(n-1)(n-2) \ldots . (n-\overline{r-1}+1)$$

consisting of $\overline{r-1}$ decreasing consecutive numbers.

Then the number of permutations of the $\overline{n-1}$ letters $b, c, d$, &c., taken $\overline{r-1}$ together, will be

$$(n-1)(n-2)\overline{n-3})\;.\;.\;.\;.\;(\overline{n-1}-\overline{r-1}+1)$$
$$\text{Or,}\quad (n-1)(n-2)\;.\;.\;.\;.\;.\;.\;.\;.\;(n-r+1).$$

And if $a$ be placed before each of these permutations, we shall have

$(n-1)(n-2)\;.\;.\;.\;.\;(n-r+1)$ permutations, each of $r$ letters in which $a$ stands first.

Similarly, it may be shown that we shall have

$(n-1)(n-2)\;.\;.\;.\;.\;(n-r+1)$ permutations, each of $r$ letters in which $b$ stands first.

and so on for all the $n$ letters $a, b, c, d$, &c.

Hence the whole number of permutations of the $n$ letters taken $r$ together will be

$$n(n-1)(n-2)\;.\;.\;.\;.\;(n-r+2)(n-r+1).$$

consisting of $r$ decreasing consecutive numbers.

It thus appears that if the formula be true for $n$ things taken $\overline{r-1}$ together, it is also true for $n$ things taken $r$ together; that is, if it be true for any one number, it is true for the next number above it.

But the formula has been proved true when $r = 3$, hence it must be true when $r = 4$; and therefore, again, it must be true when $r = 5$, and so on to any number. Hence universally

$$^nP_r \;=\; n(n-1)(n-2)\;.\;.\;.\;.\;(n-r+1). \qquad (3.)$$

If all the things be taken in every permutation, then $r = n$; an we have

$$^nP_n \;=\; n(n-1)(n-2)\;.\;.\;.\;.\;3.2.1,$$

which is generally written $\underline{|n}$. and is read "Factorial $n$."

**163.** In the last Article, the $n$ things were supposed to be all different; when any number of the things are alike, the number of permutations will be less, and may be found as follows:—

Of the $n$ things, let $p$ be of one sort, all alike, $q$ of another sort, all alike, $r$ of another, &c.

Consider, first, one of the permutations of $n$ things, of which $p$ are alike, and the rest different.

Now, if these $p$ things were all different, they would make $\underline{|p}$ permutations in place of one, the order of the other things remaining unchanged; and the same will be the case with every one of the permutations, so that the number of permutations of $n$ things, when $p$

of them are alike, equals the number when they are all different, divided by $\underline{|p}$.

$$= \frac{\underline{|n}}{\underline{|p}}.$$

Again, if there be another group of $q$ things all alike, the number of permutations equals the number of permutations when only $p$ are alike divided by $\underline{|q}$; that is, when there are $p$ of one sort and $q$ of another sort, the number of permutations of $n$ things taken all together

$$= \frac{\underline{|n}}{\underline{|p} \cdot \underline{|q}}.$$

If there be more groups of like things, the same reasoning may be repeated.

Hence, if there be $n$ things, and $p, q, r, s$, &c., denote the number of things alike in each group, the whole number of permutations of the $n$ things taken altogether

$$= \frac{\underline{|n}}{\underline{|p} \cdot \underline{|q} \cdot \underline{|r} \cdot \underline{|s}, \text{ \&c.}}. \qquad (4.)$$

### 164. To find the number of combinations of $n$ things taken $r$ together.

Every combination of $r$ things will form $\underline{|r}$ permutations, when taken all together. Hence the whole number of combinations of $n$ things taken $r$ together, equals the number of permutations of $n$ things taken $r$ together, divided by $\underline{|r}$.

Or, $${}^nC_r = \frac{{}^nP_r}{\underline{|r}}$$

$${}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}. \qquad (5.)$$

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### Examples worked out.

1. From a company of $a$ men a guard of $b$ men is chosen every night: find 1°, how many different guards can be posted; 2°, on how many nights will a particular man be engaged; 3°, on how many nights will the same $c$ given men be engaged. $c < b$?

1°. The number of guards $= \dfrac{a(a-1)\ldots(a-b+1)}{\underline{|b}}$ by (5.)

2°. If the particular man be left out, the number of different guards of $(b - 1)$ each that can be made out of the remaining $(a - 1)$ men will be

$$(a-1)(a-2)\ldots\ldots(a-b+1).$$

If, now, the excepted men be put with each one of these, each guard will consist of $b$ men; hence the number required will be

$$(a-1)(a-2)\ldots\ldots(a-b+1).$$

The third part is solved in a similar way, excepting $c$ men instead of one man; and combining the remaining men $(b - c)$ together.

2. Find the number of permutations of the letters of the word "*constitution*" taken altogether.

$$\text{The number} = \frac{\underline{|12}}{\underline{|2}\,\underline{|2}\,\underline{|3}\,\underline{|2}}. \qquad \text{by (4.)}$$

3. Find the number of combinations, also the number of permutations of the letters of the word "*infinity,*" taken seven together.

Since either of the five letters $i, n, f, t, y$ may be omitted, the number of combinations $= 5$.

And the number of permutations

$$= \underline{|7}\left\{\frac{1}{\underline{|2}\,\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|2}\,\underline{|3}} + \frac{1}{\underline{|2}\,\underline{|3}} + \frac{1}{\underline{|2}\,\underline{|3}}\right\}$$

$$= \underline{|7}\left(\frac{1}{4} + \frac{1}{6} + \frac{3}{4}\right)$$

$$= 5880.$$

*Note.*—Each of the 5 terms within the bracket represents the permutations that can be made out of each of the different combinations.

4. Find the number of combinations which can be made of $(a + b)$ things taken $r$ together, $a$ of the things being of one sort all alike and $b$ of another sort all alike.

This problem may be seen to be the same as that of finding the coefficient of $x^r$ in the product of the two factors following:

$$(1 + x + x^2 + \ldots + x^{a-1} + x^a) \times (1 + x + x^2 + \ldots + x^{b-1} + x^b)$$

There are three cases of this problem

1°. When $r \left.\begin{array}{l} < a \\ < b \end{array}\right\}$

In this case, the product $x^r$ will be formed by combining the term

$x^r$ in the first factor with 1 in the second factor

Also $x^{r-1}$ " " $x$ " "

$x^{r-2}$ " " $x^2$ " "

and so on till we come down to

1 in the first factor with $x^r$ in the second.

The whole number of times that $x^r$ occurs is

$$(r+1).$$

2°. When $$\left.\begin{array}{l} r > a \\ \phantom{r} < b \end{array}\right\}$$

The number of combinations in this case will be found to be

$$(a+1).$$

3°. When $$\left.\begin{array}{l} r > a \\ \phantom{r} > b \\ \phantom{r} < a+b \end{array}\right\}$$

In this case the number of combinations is

$$(a+b+1-r).$$

The method used in this example may be applied to some of those which follow. After the student has read the Binomial Theorem, he will find it a useful method to apply to some very complicated problems in combinations.

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## Examples LXX.

1. Find the number of permutations of 24 things taken 2 together; also 22 together; also of 6 things taken all together.

2. How many different numbers can be expressed by the figures 1, 2, 3, 4, 5, 6: find the sum of all the numbers thus formed.

3. Find the number of permutations of the letters of the word "Proportional," taken all together; also taken 6 together.

4. How many different arrangements can be made of a party of 8 persons at a round table, so that every person shall not have the same neighbours in any two of the arrangements?

In how many different ways may the party be seated at the table, every person having the same neighbours in every arrangement?

5. In how many different ways can 8 persons be seated on one side of a rectangular table? How many changes may be made in the arrangement of a crew of an eight-oared boat, none of the crew changing sides?

6. In how many different ways can 10 persons (5 ladies and 5 gentlemen) be arranged at a dinner table, so that the host and hostess may always keep their places at the ends of the table, and ladies and gentlemen may always sit alternately?

7. In how many different orders may the letters *a*, *b*, *c*, *d*, *e*, *f*, *g* be written, *g* being always the last letter?

8. How many combinations may be made of the letters of the alphabet, taking 6 consonants and 2 vowels in every combination? How

many different words may be formed, each having 6 consonants and 2 vowels?

9. How many words of 3 letters can be formed out of the English alphabet, each word having a vowel between 2 consonants? (*w* and *y* consonants.)

10. How many different parties, each of 4 ladies and 4 gentlemen, can be made out of 5 ladies and 5 gentleman? Also, find in how many of these parties a specified person can be present.

11. How many triangles can be formed by joining the angular points of a polygon of $n$ sides.

Ex. When the figure is a hexagon.

*Note.* Consider only those triangles which have all their angular points on the angles of the polygon.

12. How many quadrilateral figures may be formed by joining the angular points of an octagon?

13. How many different companies, of 50 men each, can be formed out of a regiment of 1000 men? Of how many companies will each man form a part?

14. If a certain number of things be taken 3 together, the number of combinations is 969; if they be taken 5 together, there will be 12 times as many: find the number of things.

15. Find the sum of all the different numbers which can be formed of $n$ digits $a$, $b$, $c$, &c.; the number of digits of each kind being $p$, $q$, $r$, &c. respectively, each number containing all the $n$ digits.

16. A cube is cut by 3 systems of planes, each system being parallel to a pair of opposite faces of the cube; if the number of planes in each system be $n$, find, 1st, how many different sizes of cubes will be formed; 2nd, how many different sizes of parallelopipeds.

17. If, in the last question, the number of cutting planes in each system be $n$, $p$, $q$, respectively, find the whole number of solids that will be formed.

18. Prove the following formula—

$${}^nP_r = (n - r + 1)\,P_{r-1} \qquad (1)$$

$${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} \qquad (2)$$

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \dots + {}^{r-1}C_{r-1}. \qquad (2)$$

19. If $n$ things be taken $r$ together, if $n$ be even, the number of combinations is greatest when $r = \frac{1}{2}n$, if $n$ be odd.

20. If $n$ points in a plane be joined in all possible ways by indefinite straight lines, then, if no two of the straight lines be parallel, and no three of them pass through the same point, find the whole number of points of intersection.

21. If the product of the factors $(1 + a)(1 + b)(1 + c)$, &c. to $n$ terms, find the number of terms which involve two, three, ... $n$ letters respectively. Hence, find the coefficient of $a^4$ in the expansion of $(1 + a)^n$.

22. Find the number of combinations of $(a + b + c)$ things taken $r$ together, $a$ of the things being of one sort, $b$ of another, and $c$ all different.

23. There is an army consisting of 10 European and 20 Sepoy regiments, and it is required to divide it into three detachments of 8, 10, and 12 regiments respectively, and such that the number of Europeans in any detachment shall never be less than one-fourth of the whole number of men in the detachment: find in how many ways this may be done.

# CHAPTER XVI.

## THE BINOMIAL THEOREM.

**165.** In such expressions as $(1 + x)^3$, $(a - b)^{20}$, $(3ab - xy)^{\frac{1}{2}}$, $\left(\frac{x^{\frac{1}{2}}}{3} + \frac{y^{\frac{1}{3}}}{2}\right)^{-\frac{5}{6}}$ &c., the index denotes that a certain operation is to be performed upon the Binomial within the brackets; when this operation is completed, the Binomial is said to be *expanded*.

Thus $1 + 3x + 3x^2 + x^3$ is called the *expansion* of $(1 + x)^3$.

The "Binomial Theorem" is a formula for finding the terms in the expansion of a Binomal, without multiplication or extraction of roots.

It is one of the discoveries of Sir Isaac Newton.

**166.** To prove that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3, \text{ \&c.,}$$

when $n$ is a positive integer.

If the continued product of the $n$ binomial factors

$$(1 + a)(1 + b)(1 + c) \; \ldots \; (1 + k) 1 + l),$$

be taken, we shall have a result of the following form :—

$$\begin{array}{lll} & \quad 1. & \quad 2. \\ \left\{\begin{array}{l} + a + b + c +, \&c. \;.\;.\;.\;.\;.\; + k + l \\ + ab + ac + ad \;.\;.\;.\;.\;.\;.\; + bc + bd + be + \;.\;.\;.\; + cd + ce, \&c. \\ + abc + abd \;.\;.\;.\;.\;.\;.\;.\;.\; + acd + ace + \;.\;.\;.\; + bcd + bce, \&c \\ + abcd + abce + \;.\;.\;.\;.\;.\; + bcde + bcdf + \;.\;.\;. \\ \quad \&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right. \end{array}$$

It is evident:—That the first term is unity, for it results from the multiplication of $n$ factors, each of which is unity.

That the second line contains the $n$ letters $a, b, c, d$, &c., for each term results from the multiplication of $n$ factors, $(n-1)$ of which are unity, and the other is a single letter; there are therefore as many terms as there are different letters.

That the third line contains all the combinations that can be made of the $n$ letters $a, b, c, d$, &c., taken two together, for each term results from the multiplication of $n$ factors, $(n-2)$ of which are unity, and the other two are some combination of the $n$ letters taken two together.

And generally, that where each term contains $r$ letters, the number of terms in that line must be the number of combinations of $n$ things taken $r$ together, or

$$\frac{n(n-1)(n-2)\;.\;.\;.\;(n-r+1)}{1\;.\;2\;.\;3\;.\;.\;.\;.\;\;r}.$$

Also the last line contains one term, for it is the product of the $n$ letters, and there is only one combination of $n$ things taken altogether. Now the number of terms in each line will be the same, if $a, b, c, d$, &c. be all equal to one another.

Let $\qquad a = b = c = d, \&c. = x$ suppose.

Each of the factors to be multiplied becomes $1 + x$, and the product of the $n$ factors is represented by $(1+x)^n$, and the above lines of products become respectively

$$\begin{array}{ll} \quad 1. & \\ + x + x +, \&c. \text{ to } n \text{ terms} & = nx, \\ + x^2 + x^2 +, \&c. \text{ to } \dfrac{n(n-1)}{1.2} \text{ terms} & = \dfrac{n(n-1)}{1.2}x^2, \\ + x^3 + x^3 +, \&c. \text{ to } \dfrac{n(n-1)(n-2)}{1.2.3} & = \dfrac{n(n-1)(n-2)}{1.2.3}x^3, \\ \quad \&c. \qquad\qquad \&c. & \qquad \&c. \end{array}$$

$$+ x^r + x^r, \&c. \text{ to } \frac{n(n-1)\;.\;.\;.\;(n-2+1)}{1.2.3.2} \text{ terms}$$

$$= \frac{n(n-1)\;.\;.\;.\;n-2+1)}{1.2\;.\;.\;.\;.\;r}x^2.$$

Hence

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \dots + \frac{n(n-1)\dots n-r+1)}{1.2\dots r}x^r \dots$$
$$+ nx^{n-1} + x^n. \qquad (1.)$$

$$\text{Cor. } (a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n$$
$$= a^n\left(1+\frac{x}{a}\right)^n$$
$$= a^n\left(1+n.\frac{x}{a}\right)^r + \frac{n(n-1)}{1.2}\left(\frac{x}{a}\right)^2 + \dots$$
$$+ \frac{n(n-1)\dots n-2+1}{1.2\dots r}\left(\frac{x}{a}\right)^2$$

$$\text{Or } (a+x)^n = a^n + n.a^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \dots$$
$$+ \frac{n(n-1)\dots(n-r+1)}{1.2\dots r}a^{n-2}x^r$$
$$\dots + nax^{n-1} + x^n. \qquad (2.)$$

*Note.*—In formulæ (1) and (2) the following points should be observed and remembered :—

1°. The number of terms in the expansion is one more than the index.

2°. The index of $x$ (the second term of the Binomial) increases by one in every successive term of the expansion.

3°. The coefficients of the terms equidistant from the beginning and end are alike.

4°. The last factor in the denominator of each coefficient is the same as the index of $x$ in that term.

5°. In formula (2) the sum of the indices of $a$ and $x$ in any term is equal to $n$, the original index.

Some of the most useful applications of this theorem will be seen in the examples worked out below.

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## Examples worked out.

1. Expand $(a + 3x)^6$.

By (2): $$(a+3x)^6 = a^6 + 6a^5.(3x) + \frac{6.5}{1.2}a^4(3x)^2 + \frac{6.5.4}{1.2.3}a^3.(3x)^3 \dots$$
$$\dots + 6a(3x)^5 + (3x)^6.$$
$$= a^6 + 18a^5x + 135a^4x^2 + 540a^3x^3 + 1215a^2x^4$$
$$+ 1458ax^5 + 729x^6.$$

2. Find the 6th term in the expansion of $(a^2 - b^3)^{12}$.

By (2) the 6th term is seen to be

$$\frac{12.11.10.9.8}{1.2.3.4.5}(a^2)^7.(-b^3)^5,$$

Or, $$-792a^{14}.b^{15}.$$

*Note.*—In expanding $(a^2 - b^3)^{12}$ it should be written in the form $\{a^2 + (-b^3)\}^{12}$.

3. What is the coefficient of $x^8$ in the expansion of $(1 + x^2)^{10}$?

The term involving $x^8$ is the 5th term, and the coefficient of this is

$$\frac{12.11.10.9}{1.2.3.4},$$

Or, $$99.$$

4. Find the $r$th term, also the $(n + 1)$th term in the expansion of $\left(3x - \frac{1}{3x}\right)^{2n}$.

$$\text{The } r^{\text{th}} \text{ term} = \frac{2n(2n-1)\ .\ .\ .\ (2n-r+2)}{1.2\ .\ .\ .\ .\ (r-1)}.(3x)^{2n-2+1}\left(-\frac{1}{3x}\right)^{r-1},$$

$$= (-1)^{r-1}.\frac{2n(2n-1)\ .\ .\ .\ (2n-r+2)}{1.2\ .\ .\ .\ .\ (r-1)}(3x)^{2n-2r+2}.$$

The $(n + 1)$th term is obtained from the above expression for the $r$th term by writing $(n + 1)$ instead of $r$ throughout. Hence

$$\text{The } (n+1)^{\text{th}} \text{ term} = (-1)^n.\frac{2n(2n-1)\ .\ .\ .\ (2n-n+1)}{1.2\ .\ .\ .\ .\ n}(3x)^{(2n-2n)}$$

$$= (-1)^n.\frac{2n.(2n-1)\ .\ .\ .\ (n+1)}{1.2\ .\ .\ .\ .\ n}.$$

Since $(3x)^o = 1$.

The form may be still further reduced, for

$$\frac{2n(2n-1)\ .\ .\ .\ (n+1)}{1.2\ .\ .\ .\ .\ n} = \frac{\underline{|2n}}{\underline{|n}.\underline{|n}}$$

$$= 2^n.\underline{|n}.\frac{(2n-\ )(2n-3)\ .\ .\ .\ 3.1}{\underline{|n}.\underline{|n}}$$

$$\therefore \text{The } (n+1)^{\text{th}} \text{ term} = (-1)^n.2^n.\frac{(2n-1(2n-3)\ .\ .\ .\ 5.3.1}{\underline{|n}}.$$

*Note.*—The student should note the form $(-1)^{r-1}$; it is a convenient way of expressing that the sign of the term is positive or negative, according as $r$ is odd or even.

5. Find the coefficient of $x^3$ in the expansion of $(1 + x + x^2)^{10}$.

$$\begin{aligned} \text{Now } (1 + x + x^2)^{10} &= \{1 + x(x + 1)\}^{10} \\ &= 1 + 10x(x + 1) + 45\overline{x(x + 1)}^2 \\ &\quad + 120\,\overline{x(x + 1)}|^3 +, \&c. \\ &= 1 + 10x(x + 1) + 45x^2(x + 1)^2 \\ &\quad + 120x^3(x + 1)^3, \&c. \end{aligned}$$

It is evident that all the powers of $x$ which appear after the 4th term are higher than $x^3$; we must now select the terms involving $x^3$ which arise by multiplying out the 3rd and 4th terms, and add them together. These are $45 \times 2 + 120$,

Or, $90 + 120$;

Or, 210, the coefficient of $x^3$.

6. Find the number of terms in the expansion of $(a + b + c)^{18}$.

$$\begin{aligned} \text{Now } (a + b + c)^{18} &= (a + \overline{b + c})^{18} \\ &= a^{18} + 18a^{17}(b + c) + \frac{18.17}{1.2}a^{16}(b + c)^2 \\ &\quad + \frac{18.17.16}{1.2.3}a^{15}(b + c)^3, \&c. \end{aligned}$$

The successive powers of $(b + c)$ are now to be expanded, and the terms of each expansion are to be multiplied into the corresponding power of $a$ outside the bracket. It is evident the whole number of terms will be

$$1 + 2 + 3 + 4 + \ldots\ldots + 19,$$

Or, $$(2 + 18)\frac{10}{2} = 190.$$

7. Find the greatest coefficient in the expansion of $(3a + 2x)^{30}$.

The coefficient of the $r^{\text{th}}$ term of this is

$$\frac{30.29\ 28 \ldots\ldots (30 - r + 2)}{1.2.3 \ldots\ldots (r - 1)},$$

and the factor which is introduced to form the coefficient of the $\overline{r + 1}|^{\text{th}}$ term is $\frac{31 - r}{r}$.

Now when $r$ has such a value that $\frac{31 - r}{r}$ is either equal to or less than 1, the coefficients from this point cease to increase, or begin to decrease. To find the term which has the greatest coefficient we therefore put

$$\frac{31 - r}{r} < 1$$

Or, $$31 - r < r$$

Or, $$31 < 2r$$

That is $\quad 2r > 31$

Or, $\quad r > 15\frac{1}{2}$

$\therefore \quad r = 16.$

gives the greatest coefficient, and the value of it is 155117520.

8. Find the sum of the coefficients of an expanded Binomial.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 +, \&c.$$

Let $x = 1$.

Then $\quad 2^n = 1 + n + \dfrac{n(n-1)}{1.2} +, \&c.$

The sum required is therefore $2^n$.

By a similar method it may be proved that the sums of the coefficients of the *odd* and *even* terms respectively are alike.

9. If 1, $a$, $b$, $c$, &c., be the cofficients, taken in order, of the expansion of $(1+x)^n$, show that when $n$ is any positive integer greater than 1, $\quad 1 - 2a + 3b - 4c + \&c. \ldots = 0.$

For $\quad (1-x)^n = 1 - nx + \dfrac{n(n-1)}{1.2}x^2 - \dfrac{n(n-1)(n-2)}{1.2.3}x^3, \&c.$

Also, $\quad nx(1-x)^{n-1} = nx - n(n-1)x^2 + \dfrac{n(n-1)(n-2)}{1.2}x^3, \&c.$

By Subtraction, $\quad (1-x)^n - nx(1-x)^{n-1}$

$$= 1 - 2nx + 3\frac{n(n-1)}{1.2}x^2 - 4\frac{n(n-1)(n-2)}{1.2.3}x^3 +, \&c.$$

$$= 1 - 2ax + 3bx^2 - 4cx^3 + \&c.$$

The student should carefully work out the coefficient of $x^r$; it ought to be $\quad (r+1)\dfrac{n(n-1)\ldots(n-r+2)}{\underline{|r-1}}$.

Put $x = 1$ in the above equation, the required relation will result.

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## Examples LXXI.

1. Expand $(a+x)^7$; $(1+5b)^9$; $(2a^2-3x^2)^8(x^2-y^2)^{12}$; $\left(a-\dfrac{b}{2}\right)^{13}$; $\left(\dfrac{3}{5}a-1\right)^6$; $\left(\dfrac{ab}{2}-\dfrac{2}{ab}\right)^{11}$; $\left(a^{\frac{1}{2}}-x^{\frac{1}{3}}\right)^{n^2+1}$; $\left(1-\dfrac{3}{2}x\right)^{2n}$.

2. Write down the $4^{th}$ term of $(1+x)^{12}$; the $5^{th}$ of $(a^2+b^2)^p$; the $n^{th}$ term of $(x+a)^{\overline{n+8}}$; the $\overline{p+1|}^{th}$ of $(x+y)^{2p+1}$; the $\overline{n+1|}^{th}$ of $\left(\dfrac{a}{x}+\dfrac{x}{a}\right)^{2n}$; the middle term of $\left(ax+\dfrac{a}{x}\right)^{4p}$.

3. Find the coefficient of $x^{12}$ in $(1-x)^{13}$; of $x^8$ in $(a+x)^{2p}$; of $x^2$ in $\left(\frac{x}{x}-\frac{a}{x}\right)^p$; of $x^r$ in $(2x^2-4ax)^{n+4}$; of $x^{12}$ in $\left(x^{\frac{1}{3}}x^{\frac{2}{3}}-b\right)^{20}$.

4. Find the term involving $a^{17}.b^7$ in $(3a-2b)^{24}$; $a^{\overline{3n-10}}.b^{\overline{2n-6}}$ in $(a^3b^2-ab)^{n-2}$; the term independent of $x$ and $y$ in $\left(\frac{x}{y}-\frac{y}{x}\right)^{4n}$; and in $\left(y^2-\frac{c}{y}\right)^{12}$.

5. Find the number of terms in the expansion of $(a+b+c)^{12}$, and in that of $(3a-2b+4c)^n$.

6. Find the coefficient of $x^{12}$ in the expansion of $(1+x+x^2+x^3)^5$, and of $x^{20}$ in $(1-x^2+x^4-x^6)^{12}$.

7. What is the sum of the coefficients in the expansion of $(a+b+c)^{12}$ and $(3a-2b-4c)^n$ ?

8. Expand $(1+\sqrt{x^2-1})^{12}+(1-\sqrt{x^2-1})^{12}$ in powers of $x$; also find the coefficient of $x^{\frac{5}{2}}$ in $(1+\sqrt{x})^{12}-(1-\sqrt{x})^{12}$.

9. Find the sum of the series

$$\frac{1.3.5\ldots.(2r-3)}{\underline{|r-1}}+\frac{1.3.5\ldots.(2r-5)}{\underline{|r-2}}.\frac{3}{1}+\frac{1.3.5\ldots.(2r-7)}{\underline{|r-3}}.\frac{3.5}{1.2}+,\ \&c.$$

10. The coefficient of the 4th term of $(1+x)^n$ : that of the 6th term : : 2 : 3. Find $n$.

11. If 1, $a_1$, $a_2$, $a_3$, &c. be the coefficients of the expansion of $(a+x)^n$ ; find the value of $1+a_1+a_2+$, &c.

12. Prove that the series $2.a^n+2^2.\frac{n+1}{n}.\frac{n}{1}.a^{n-1}+2^3.\frac{n+2}{n}.\frac{n(n-1)}{1.2}a^{n-2}+$, &c. $=(a+2)^{n-1}(2a+6)$.

Also, find the value of the series obtained by multiplying the successive terms in the expansion of $(1+3x)^n$ by the numbers 1, 3, 5, &c., respectively.

13. Using the notation of (11), prove that

$$1.a_m+a_1a_{m+1}+a_2a_{m+1}+\ldots\ldots+a^m a_{2m}$$
$$=\frac{2^{2m}.(4m-1)(4m-3)\ldots.3.1}{\underline{|m}}.$$

**14. Find the sum of the squares of the coefficients in the expansion of $(1 + x)^{2n}$.**

**15. The third, fourth, and fifth terms of the expanded binomial $(x + y)^n$, are $a$, $b$, $c$, respectively; find the value of $n$. Can the values of $x$ and $y$ be found from these data?**

**167.** To prove the Binomial Theorem, when the index is a negative integer, or that

$$(1 + x)^{-n} = 1 - nx + \frac{(-n)(-n-1)}{1.2} n^2$$

$$+ \frac{-n(-n-1)(-n-2)}{1.2.3} x^3 +, \&c.$$

Or, $$(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{1 \cdot 2} x^2$$

$$- \frac{n(n+1)(n+2)}{1.2.3} x^3 +, \&c.$$

Now $(1 + x)^{-1} = \dfrac{1}{1 + x}$.

And, by actual division,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 +, \&c. \textit{ ad inf.}$$

Again, $(1 + x)^{-2} = (1 + x)^{-1} (1 + x)^{-1}$

$$= \frac{1}{1+x} \cdot (1 - x + x^2 - x^3 +, \&c.)$$

$$= 1 - 2x + 3x^2 - 4x^3 +, \&c. \text{ by actual division,}$$

Or, $$(1 + x)^{-2} = 1 - \frac{2}{1} \cdot x + \frac{2.3}{1 \cdot 2} x^2 - \frac{2.3.4}{1.2.3} x^3 +, \&c.$$

And, it may again be proved by actual division, that

$$(1 + x)^{-3} = 1 - \frac{3}{1} x + \frac{3.4}{1.2} x^2 - \frac{3.4.5}{1.2.3} x^3 +, \&c.$$

Suppose, now, that the law of the coefficients, which is here observed, holds good for all values of the index from $-1$ to $-(x - 1)$; so that

$$(1 + x)^{-\overline{n-1}} = 1 - (n - 1)x + \frac{(n-1)n}{1.2} x^2 - \frac{(n+1)(n-1)}{1.2.3} x^3$$

$$+, \&c.$$

Now $$(1 + x)^{-n} = (1 + x)^{-\overline{n-1}} (1 + x)^{-1}$$

$$= \frac{1}{1+x} \left\{ 1 - (n - 1)x + \frac{(n-1)n}{1.2} x^2 - \right.$$

$$\left. \frac{(n-1)n(n+1)}{1.2.3} x^3 +, \&c. \right\}$$

And if the quantity within the brackets be divided by $(1+x)$, we shall have for a quotient

$$1 - nx + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 +, \&c.$$

If the division be continued far enough, we shall have the following remainder

$$\frac{-(n-1)(-n)\,..\,(-n-r+2)}{1.2\;\ldots\ldots\;r}x^r + \frac{-n(-n-1)\,..\,(-n-r+2)}{1.2\;\ldots\;(r-1)}x^r$$

which equals $\dfrac{(-u)(-n-1)\;\ldots\;(-n-r+1)}{1.2\;\ldots\ldots\ldots\;r}x^r.$

And this will be the coefficient of $x^r$ in the quotient, and it is of the same form as the coefficient of $x^r$ in $(1+x)^{-\overline{n-1}}$.

Hence, then, it appears, that if the law hold good for an index $-(n-1)$, it also holds for an index $-n$; that is, if it be true for any one value of the index, it is also true for the next higher value; but it was shown to be true by actual division when the index was $-3$. It must therefore be true when the index is $-4$; and since it is true when $n = -4$, it must be true when $n = -5$, and by continuing this reasoning, it will appear to be generally true. That is, when $n$ is any negative integer,

$$(1+x)^{-n} = 1 - nx + \frac{-n(-n-1)}{1.2}n^2 + .\frac{-n(-n-1)(-n-2)}{1.2.3}x^2 +, \&c. \qquad (3)$$

This is the same as formula (1), if $-n$ be written instead of $n$.

If the student will write down a large number of the terms of this expansion, he will see that the signs of the terms will be alternately + and −, and that the formula may be conveniently written thus

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + \frac{(-1)^r n(n+1)\ldots(n+r-1)}{1.2\;\ldots\ldots\;2}x^r, \&c. \qquad (4)$$

Also,

$$\begin{aligned}(a+x)^{-n} &= \left\{a\left(1+\frac{x}{a}\right)\right\}^{-n}\\ &= a^{-n}\left(1+\frac{x}{a}\right)^{-n}\\ &= a^{-n}\left(1-n\frac{x}{a}\right) + \frac{n(n+1)}{1.2}\left(\frac{x}{x}\right)^2 - \frac{n(n+1)(n+2)}{1.2.3}\left(\frac{x}{a}\right)^3, \&c.\end{aligned}$$

$$= a^{-n} - na^{-\overline{n+1}}x + \frac{n\,\overline{n+1}}{1.2}a^{-\overline{n+2}}x^2 - \dots$$

$$+ (-1)^r\frac{n\,\overline{n+1}\dots\overline{n+r-1}}{1.2\dots r}a^{-\overline{n+r}}x^r, \&c.$$

$$(a-x)^{-n} = a^{-n} + na^{-\overline{n+1}}x + \frac{n\,\overline{n+1}}{1.2}a^{-\overline{n+2}}x^2 +, \&c. \quad (3)$$

Note. In the last two expansions the following points should be observed and remembered:—

1°. The number of terms in the expansion is indefinitely great, because no factor in any of the coefficients ever becomes zero.

2°. The sum of the indices of $a$ and $x$ in any term is $(-n)$.

3°. When the sign of the index, and the sign which connects the two terms of the Binomial are alike, all the terms of the expansion are positive; when they are different, the terms of the expansion are alternately positive and negative.

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## Examples LXXII.

1. Expand $(1+x)^{-3}$, $(a-b)^{-2}$, $\frac{1}{a-b}$, $\frac{1}{(1-a)^2}$, $\frac{1}{(a-b)^{\frac{1}{2}}}$.

2. Find the 8th term in the expansion of $(a^2-b^2)^{-4}$, the $\overline{p+1}|^{th}$ term in $(1-a)^{-2p}$, and the coefficient of $x^n$ in $\frac{1}{(a+x)^p}$.

3. Expand $(x-c)^{-6}$, $\frac{1}{(a^{\frac{1}{3}}-x^{\frac{1}{3}})^3}$, $\left(\frac{x}{2}-\frac{2}{x}\right)^{-m}$, $\frac{1}{a^{\frac{2}{3}}+(bc)^{\frac{1}{3}}}$, $\frac{1}{\left(x-\frac{1}{x}\right)^{2n+1}}$, and write down the $r^{th}$ term of each.

4. Find the coefficient of $x^6$ in $(1+x+x^2+, \&c., \textit{ad inf.})^p$; also of $x^{2p}$ in $(1+2x+3x^2+, \&c., \textit{ad inf.})^{-1}$; also the term involving $\frac{x^6}{a^{14}}$ in $(a^2-x)^{10}$.

5. Find the sum of the series

$$1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}, \&c., \textit{ad inf.}$$

$$\text{Of } \frac{1}{2}\left(\frac{3.4}{3^2}+\frac{4.5}{3^3}+\frac{5.6}{3^4}+, \&c. \dots \textit{ad inf.}\right);$$

Of $1 - 2 + \frac{8}{3} - \frac{40}{9} + \frac{80}{27}$, &c., *ad inf.*

6. Find the sum of the infinite series, whose $r^{\text{th}}$ is $r.\left(\frac{x}{1+x}\right)^{r-1}$.

Also of the infinite series, whose $\overline{2p+1}|^{\text{th}}$ term is $(p+1)(2p+1)\frac{(4x^5)^p}{\left(1+x^{\frac{5}{2}}\right)^{2p}}$.

7. Find the greatest term in the expansion of $\left(\frac{2}{3} + \frac{3x}{2}\right)^{-9}$; $\left(1 - \frac{1}{5}\right)^{-2}$; $(1-x)^{-\frac{3}{5}}$ when $x = \frac{20}{19}$.

8. Expand $\left(\frac{1+x}{1-x}\right)^n$; $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$; and find to four terms the $m^{\text{th}}$ root of $\frac{1+x^2}{1+2n^2}$.

9. The fourth term of $(1-x)^n$ : the fourth term of $(1+x)^n$ : : 5 : 1. Find $n$.

10. The seventh term of $(1-x)^{-7}$: the fifth term : : 5 : 11. Find $x$.

11. Show that $\frac{q(q+1)\ \ldots\ (q+r-1)}{\underline{|r}} + \frac{p}{1}.\frac{q(q+1)\,..\,(9+2-2)}{\underline{|r-1}}$ $+ \ldots + \frac{p(p-1)}{1.2}.\frac{q(q+1)\ \ldots\ (q+r-3)}{\underline{|r-2}}$, &c., to $(r+1)$ terms $= \frac{(p+q)(p+q-1)\ \ldots\ (p+q+r-1)}{\underline{|r}}$.

12. If ${}^pC_q$ denote the coefficient of $x^q$ in the expansion of $(1+x)^p$, prove that

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \ldots + {}^{r-1}C_{r-1}. $$

Also that $\frac{{}^nC_{r-1} + {}^nC_r}{{}^nC_{r-1}}.r$ is constant for all values of $r$.

13. Show that the coefficient of $x^n$ in the expansion of $\frac{1}{(1-ax)(1-a^2x)}$ equals $a^n\frac{1-a^{n+1}}{1-a}$; also that the coefficient of $x^n$ in the expansion of $\frac{1}{(1-ax)(1-a^2x)(1-a^3x)}$, where the number of factors is unlimited, and $a < 1$ equals

$$ \frac{a^n}{(1-a)(1-a^2)\ \ldots\ .\ (1-a^n)}. $$

14. Find the cube root of 29 to four places of decimals, and the value of $\sqrt[5]{\frac{1000}{999}}$ to twelve places; both by the Binomial Theorem.

15. Find the limiting value of $\left(1 + \frac{x}{n}\right)^n$, when $n$ is indefinitely increased.

16. If $x$ be greater than $y$, prove that $x^p - y^p$ lies between $px^{p-1}(x - y)$ and $py^{p-1}(x - y)$.

17. Find the value of $\frac{\sqrt{2}+1}{\sqrt{2}-1}$, and of $\frac{1}{(\sqrt{3}+1)^{-3} + (\sqrt{3}-1)^{-3}}$ to 6 places of decimals by the Binomial Theorem.

**168.** Before giving the proof of the Binomial Theorem when the index is fractional, it will be necessary to discuss the following preliminary proposition.

Prop. If $A_0 + A_1 x + A_2 x^2 + \ldots\ldots + A_r x^r \quad (1)$

And $B_0 + B_1 x + B_2 x^2 + \ldots\ldots + B_r x^r \quad (2)$

be two series, in which $A_0$, $A_1$, and $B_0$, $B_1$, &c., are constant, and $x$ a variable, which may receive any value whatever, and if these two series be always equal to each other,

Then shall $A_0 = B_0$, $A_1 = B_1 \ldots\ldots A_r = B_r$.

The equality may be stated in the form

$$(A_0 - B_0) + (A_1 - B_1)x + (A_2 - B_2)x^2 \ldots + (A_r - B_r)x^r = 0.$$

Since $x$ may have any value, we may give $x$ a value so small that $(A_1 - B_1)x + (A_2 - B_2)x^2 +$, &c., shall be less than any assignable quantity; hence also $(A_0 - B_0)$ must be less than any assignable quantity,

That is $A_0 = B_0$.

Then $(A_1 - B_1)x + (A_2 - B_2)x^2 + \ldots + (A_r - B_r)x^r = 0.$

Dividing this by $x$, we have

$$(A_1 - B_1) + (A_2 - B_2)x + \ldots\ldots + (A_r - B_r)x^{r-1} = 0.$$

And for the same reason as before, we have

$$A_1 = B_1.$$

By continuing this process, we shall get

$$A_2 = B_2;\ A_3 = B_3, \text{ \&c.} \ldots A_r = B_r.$$

**169.** To prove the Binomial Theorem for a fractional index.

Let us assume that

$$(1+x)^{\frac{p}{q}} = 1 + a_1x + a_2x^2 + \ldots . + a_rx^r, \text{ \&c.}$$

And $$(1+x)^{\frac{p}{q}-1} = 1 + b_1x + b_2x^2 + \ldots . + b_rx^r, \text{ \&c.}$$

Where $p$ and $q$ are both positive integers, and $b_1$, $b_2$, &c., are the values which $a_1$, $a_2$, &c., take when $\left(\frac{p}{q} - 1\right)$ is written for $\frac{p}{q}$.

Let $$x = y + z$$

$$\begin{aligned} \therefore \quad (1+x)^{\frac{p}{q}} &= (1+y+z)^{\frac{p}{q}} \\ &= 1 + a_1(y+z) + a_2(y+z)^2 + \ldots + a_r(y+z)^r, \text{ \&c.} \\ &= 1 + z(a_1 + 2a_2y + 3a_3y^2 + \ldots + ra_ry^{r-1}) \\ &\qquad + \text{higher powers of } y \text{ and } z. \qquad (1) \end{aligned}$$

Again,

$$\begin{aligned} (1+x)^{\frac{p}{q}} &= (1+y+2)^{\frac{p}{q}} = (1+y)^{\frac{p}{q}} \cdot \left(1 + \frac{z}{1+y}\right)^{\frac{p}{p}} \cdot \\ &= (1+y)^{\frac{p}{q}}\left\{1 + a_1 \cdot \frac{z}{1+y} + a_2\left(\frac{z}{1+y}\right)^2 + \ldots . \right. \\ &\qquad \left. + a_2\left(\frac{z}{1+y}\right)^r \ldots \right\} \\ &= (1+y)^{\frac{p}{q}} + a_1z(1+y)^{\frac{p}{q}-1} + a_2z^2(1+y)^{\frac{p}{q}-2} +, \text{ \&c.} \\ &= (1+y)^{\frac{p}{q}} + a_1z(1 + b_1y + b_2y^2 + \ldots + b_ry^r \ldots) \quad (2) \\ &\qquad + \text{higher powers of } y \text{ and } z. \end{aligned}$$

To find the value of $a_1$, we have

$$\begin{aligned} (1+x)^p &= (1 + a_1x + a_2x^2, \text{ \&c.})^q, \\ \text{Or,} \quad 1 + px, \text{ \&c.} &= 1 + qa_1x \ldots, \text{ \&c.} \\ \therefore \quad p &= qa_1 \qquad \text{Art. (168).} \\ \text{Or,} \quad a_1 &= \frac{p}{q}, \\ \text{And} \quad b_1 &= \frac{p}{q} - 1. \end{aligned}$$

If, now, we equate the coefficients of $yz$, $y^2z$, $y^3z$, &c., in (1) and (2) agreeably to the principles of Art. (168), we shall have

$$2a_2 = a_1b_1, \text{ or } a_2 = \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)}{1 \cdot 2};$$

$$3a_3 = a_1b_2, \text{ or } a_3 = \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)}{1 \cdot 2 \cdot 3};$$

And generally,

$$ra_r = a_1 b_{r-1}, \text{ or } a_r = \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{1\,.\,2\,\ldots\ldots\,r}.$$

Hence, we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}\,x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1\,.\,2}\,x^2 + \ldots$$

$$\ldots + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{|\underline{r}}\,x^r, \text{ \&c.},$$

which is the Binomial Theorem for any fractional index, positive or negative.

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## Examples worked out.

1. Find the coefficient of $x^r$ in the expansion of $\dfrac{(1-2x)^2}{\sqrt[4]{1-3x}}$.

$$\text{Now } (1-2x)^2(1-3x)^{\frac{1}{4}} = (1-4x+4x^2)\,.\,(1-\tfrac{1}{4}\,.\,3x+, \text{ \&c.}$$
$$+\frac{(-1)^r\,.\,\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\ldots\left(\frac{1}{4}-r+1\right)}{|\underline{r}}\,.\,(3x)^r +, \text{ \&c.})$$

It is evident that to obtain the coefficient of $x^r$, in the product of these two expansions, we must select the coefficients of $x^{r-2}$, $x^{r-1}$, $x^r$ from the second expansion, and multiply these into the numbers 4, − 4, and 1 respectively. These products, when written in their simplest form, will be

$$-4\,.\,\frac{1.3.7\ldots(4r-13)}{4^{r-2}\,.\,|\underline{r-2}} + 4\,.\,\frac{1.3.7\ldots(4r-9)}{4^{r-1}\,.\,|\underline{r-1}}$$
$$-\frac{1.3.7\ldots(4r-5)}{4^r\,.\,|\underline{r}}.$$

2. Approximate, by the Binomial Theorem, to the value of $\sqrt[3]{61}$.

$$\text{Now, } \sqrt[3]{61} = (61)^{\frac{1}{3}} = (64-3)^{\frac{1}{3}} = \left\{4^3\left(1-\frac{3}{4^3}\right)\right\}^{\frac{1}{3}}$$
$$= 4\left(1-\frac{3}{64}\right)^{\frac{1}{3}}.$$

$$\text{And } \left(1-\frac{3}{64}\right)^{\frac{1}{3}} = 1 - \frac{1}{3}\,.\,\frac{3}{64} - \frac{1}{9}\,.\left(\frac{3}{64}\right)^2 - \frac{5}{81}\,.\left(\frac{3}{64}\right)^3$$
$$- \frac{10}{243}\,.\left(\frac{3}{64}\right)^4 -, \text{ \&c.}$$

$$= 1 - \frac{1}{64} - \frac{1}{64^2} - \frac{5}{3} \cdot \frac{1}{(64)^3} - \frac{10}{3} \cdot \frac{1}{64^4} -, \&c.$$

To find the actual value of the series in decimals, we may proceed as follows:—

$$\frac{1}{64} = \cdot 015625$$

$$\frac{1}{64^2} = \cdot 015625 \div 64 = \cdot 000244140, \&c.$$

$$\frac{1}{64^3} = \cdot 000244140 \div 64$$

$$\therefore \quad \frac{5}{3} \times \frac{1}{64^3} = \cdot 000003815 \times \frac{5}{3} = 000006358, \&c.$$

$$\frac{1}{64^4} = \cdot 000003815 \div 64$$

$$\therefore \quad \frac{10}{3} \times \frac{1}{64^4} = \cdot 0000000596 \times \frac{10}{3} = \begin{array}{l} \underline{\cdot 000000198} \\ \cdot 000250696, \end{array} \&c.$$

If we stop at this point, we have

$$\begin{aligned} \sqrt[3]{61} &= 4\,(1 - \cdot 000250696) \\ &= 4 \times \cdot 999749304 \\ &= 3 \cdot 998997216, \end{aligned}$$

which may be considered true to four places.

3. Show that $\dfrac{px^p - qx^q}{p - q}$ and $x^{p+q}$ are nearly equal, when $x$ is nearly equal to 1.

Let $x = 1 + h$; $\therefore$ $h$ is very small.

And the expression becomes

$$\frac{p\left(1 + ph + \dfrac{p\,(p-1)}{p-2}\,h^2 +, \&c.\right) - q\left(1 + qh + \dfrac{q\,(q-1)}{1\,.\,2}\,h^2 +, \&c.\right)}{p - q}$$

$$= \frac{1}{p-q}\Big\{(p - q) + (p^2 - q^2)\,h + \tfrac{1}{2}\,(p^2 - q^3)\,h^2 - \tfrac{1}{2}\,(p^2 - q^2)\,h^2$$
$$+ \text{ higher powers of } h.\Big\}$$

$$= 1 + (p + q)\,h + \tfrac{1}{2}\,(p^2 + pq + q^2 - p - q)\,h^2.$$

Now, since $h$ is very small, $h^2$ is very much smaller than $h$, and all the terms after $h^2$ are still much smaller; therefore, unless $p$ or $q$ be very large, $1 + (p + q)\,h$ is nearly equal to $\dfrac{px^n - qx^q}{p - q}$; also, $1 + (p + q)\,h$ is nearly equal to the expansion of $(1 + h)^{p+q}$ or $x^{p+q}$.

Whence $\dfrac{px^p - qx^q}{p - q} = x^{p+q}$ nearly.

# CHAPTER XVII.

## THE MULTINOMIAL THEOREM.

**170.** The "Multinomial Theorem" is a formula for expanding the power of an expression which consists of any number of terms more than two, and it may be regarded as an extension of the Binomial Theorem.

Before proceeding to the general investigation, we will discuss a particular case.

Consider the expression

$$(a+b+c+d+e)^6.$$

The expression of this mignt be obtained by multiplying together the six multinomial factors written below :—

$$\begin{aligned}&(a+b+c+d+e)\\ \times\ &(a+b+c+d+e)\\ \times\ &(a+b+c+d+e)\\ \times\ &(a+b+c+d+e)\\ \times\ &(a+b+c+d+e)\\ \times\ &(a+b+c+d+e).\end{aligned}$$

Now, every term in the product of these is composed of six letters, one being taken out of each line; thus there will be a number of terms $a^2b^3d$, each containing two $a$'s, three $b$'s, and one $d$. Now the number of terms $a^2b^3d$ will be the same as the number of different ways in which we can select two $a$'s, three $b$'s, and one $d$ out of the six lines above.

And this number of ways is the same as the number of different orders of the letters *aabbbd*, taken all together; for every such order represents the formation of a term by multiplication.

That is, the whole number of terms involving $a^2b^3d$ will be the number of permutations of *six* things taken all together, when two are of one sort, all alike, three of another sort, all alike, and one of another sort.

$$\therefore \quad \text{Coefficient of } a^2b^3d = \frac{\underline{|6}}{\underline{|2}\ \underline{|3}\ \underline{|1}}.$$

**171.** We shall now proceed to find the general term in the expansion of

$$(a + b + c +, \text{ \&c., to any number of terms})^n,$$

when $n$ is a positive integer.

Now, by applying the reasoning of the former case, we see that all the terms in the product will consist of $n$ factors, and each term must be of the form $a^p.b^q.c^r$, &c., where we must have $p + q + r +, = n$.

Again, the number of terms $a^p.b^q.c^r$, &c. will be the same as the number of permutations of $n$ things taken all together, when $p$ are of one sort all alike, $q$ of another sort all alike, &c.

Hence the coefficient of the quantity $a^p.b^q.c^r$, &c., in the expansion will be

$$\frac{\underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|r}, \text{ \&c.}}\ .$$

And the general term of the expansion of $(a + b + c + d +, \text{ \&c.})^n$ is said to be

$$\frac{\underline{|n}}{\underline{|p}\,.\underline{|q}\,.\underline{|r}} \, . \, a^p \, . \, b^q \, . \, c^r \, . \, , \text{ \&c.} \qquad (1)$$

**172.** To expand a power of a Multinomial we must adopt the general form just given, and give $p$, $q$, $r$, &c., all the particular values which they can possibly receive, consistent with the relation

$$p + q + r +, \text{ \&c.} = n.$$

The terms which result from this substitution will be the complete expansion required.

**Ex.** Expand $(a+b+c)^5$.

All the possible values of $p$, $q$, $r$, are arranged in the accompanying table; and since there are 21 sets of values, the number of terms in the expansion is 21.

The first four terms of the expansion will be

$$\frac{\lfloor\underline{5}}{\lfloor\underline{5}}.a^5.b^0.c^0+\frac{\lfloor\underline{5}}{\lfloor\underline{4}\,\lfloor\underline{1}}.a^4.b^1.c^0+\frac{\lfloor\underline{5}}{\lfloor\underline{4}\,\lfloor\underline{1}}\,a^4b^0c^1$$

$$+\frac{\lfloor\underline{5}}{\lfloor\underline{3}\,\lfloor\underline{1}\,\lfloor\underline{1}}.a^3b^1c^1.$$

Or, $$a^5+5a^4b+5a^4c+20a^3bc.$$

| $p$ | $q$ | $r$ |
|---|---|---|
| 5 | 0 | 0 |
| 4 | 1 | 0 |
| 4 | 0 | 1 |
| 3 | 1 | 1 |
| 3 | 2 | 0 |
| 3 | 0 | 2 |
| 2 | 3 | 0 |
| 2 | 0 | 3 |
| 2 | 2 | 1 |
| 2 | 1 | 2 |
| 1 | 4 | 0 |
| 1 | 0 | 4 |
| 1 | 3 | 1 |
| 1 | 1 | 3 |
| 1 | 2 | 2 |
| 0 | 5 | 0 |
| 0 | 0 | 5 |
| 0 | 4 | 1 |
| 0 | 1 | 4 |
| 0 | 3 | 2 |
| 0 | 2 | 3 |

The above investigation is independent of the Binomial Theorem, but it applies only to those cases in which the index is a positive integer. Another proof of the Theorem will now be given, which holds when the index is integral or fractional, positive or negative.

**173.** To find the general term in the expansion of $(a+b+c+d+, \&c.)^n$, for any value of the index.

$$\text{Let}\qquad b+c+d+,\&c. = x;$$

$$\therefore\qquad (a+b+c+d+,\&c.)^n = (a+x)^n.$$

The general, or $\overline{p+1}|^{\text{th}}$ term of this will be

$$\frac{n(n-1)(n-2)\ldots\ldots(n-p+1)}{\lfloor\underline{p}}.a^{n-p}x^p. \qquad (1).$$

$$\text{Let}\qquad c+d+,\&c. = y;$$

$$\therefore\qquad x = b+y,$$

$$\text{And}\qquad x^p = (b+y)^p.$$

The general, or $\overline{q+1}|^{\text{th}}$ term of this will be

$$\frac{p(p-1)(p-2)\ldots(p-q+1)}{\lfloor\underline{q}}.b^{p-q}.y^q. \qquad (2.)$$

$$\begin{array}{lrl} \text{Let} & d+e+, \&c. & = z; \\ \therefore & y & = c+z, \\ \text{And} & y^q & = (c+z)^q. \end{array}$$

The general, or $\overline{r+1}|^{\text{th}}$ term of this will be

$$\frac{q(q-1)(q-2)\ldots(q-r+1)}{\underline{|r}}.c^{q-r}z^r. \qquad (3.)$$

Let this process be continued to the last term of the multinomial; it is evident that the general terms will all be of the same form.

By substituting in (1) the general term (2) of $x^p$, (1) becomes

$$\frac{n(n-1)(n-2)\ldots\ldots(n-p+1)}{\underline{|p}}.a^{n-p}$$

$$.\frac{p(p-1)\ldots(p-q+1)}{\underline{|q}}.b^{p-q}.y^q.$$

And by again substituting for $y^q$ the general term (3) of $y^q$, (1) becomes

$$\frac{n(n-1)\ldots\ldots(n-p+1)}{\underline{|p}}.\frac{p(p-1)\ldots\ldots(p-q+1)}{\underline{|q}}$$

$$.\frac{q(q-1)\ldots\ldots(q-r+1)}{\underline{|r}}.a^{n-p}.b^{p-q}.c^{q-r}.z^r.$$

Which again may be put in the form

$$\frac{n(n-1)\ldots(n-p+1)}{\underline{|p}}.\frac{\underline{|p}}{\underline{|q}\,\underline{|p-q}}.\frac{\underline{|q}}{\underline{|r}\,\underline{|q-r}}a^{n-p}.b^{p-q}.c^{q-r}.z^r.$$

$$\text{Or,}\quad \underline{n(n-1)\ldots(n-p+1).a^{n-p}}.\frac{b^{p-q}}{\underline{|p-q}}.\frac{c^{q-r}}{\underline{|q-r}}.\frac{z^r}{\underline{|r}}. \qquad (4.)$$

If $n$ be a positive integer, this may be put in the form

$$\frac{\underline{|n}}{\underline{|n-p}.\underline{|p-q}.\underline{|q-r}\ \underline{|r}}.a^{n-p}.b^{p-q}.c^{q-r}.z^r.$$

It will be seen that this is in fact the same formula as (1) in Art. (171); for the sum of the indices of $a$, $b$, $c$, &c., is $n$, and each of those indices corresponds to the number of factors in the denominator.

It must be observed that when $n$ is fractional the index of $a$ will be fractional, and if $n$ be negative, some of the indices of $a$ will be negative.

Formula (4) may be reduced to the more convenient form of (I), if we put

$$\begin{aligned} n - p &= p^1 \\ p - q &= q^1 \\ q - r &= r^1 \\ \text{\&c.} &\quad \text{\&c.} \end{aligned}$$

It then becomes

$$n(n-1)(n-2)\ldots\ldots(p'+1)\,\frac{a^{p'}}{\underline{|p'}}\cdot\frac{b^{q'}}{\underline{|q'}}\cdot\frac{c^{r'}}{\underline{|r'}}\ldots$$

And if we suppress the accents, we have the general term of *any* multinomial represented by

$$n(n-1)n-2)\ldots(p+1)\,.\,\frac{a^p\,.\,b^q\,.\,c^r}{\underline{|q}\,\underline{|r}},\ \text{\&c.} \qquad \text{(II.)}$$

where $p + q + r$, &c. $= n$.

And all the indices except $p$ are positive integers.

**174.** Let the Multinomial to be expanded be

$$(a + bx + cx^2 + dx^3 +, \text{\&c.})^n.$$

When $n$ is a positive integer, the general term is

$$\frac{\underline{|n}\,.\,a^p\,.\,(bx)^q\,(cx^2)^r\,(dx^3)^s\ \ldots\ldots\ \text{\&c.}}{\underline{|p}\ \underline{|q}\ \underline{|r}\ \underline{|s}}$$

and the index of $x$ is seen to be

$$q + 2r + 3s +, \text{\&c.}$$

Hence, if it be required to find all the terms which involve a particular power of $x$, as $x^m$, we must give $p, q, r$ such values as will satisfy the two equations

$$p + q + r +, \text{\&c.} = n \qquad (1)$$

and

$$q + 2r + 3s +, \text{\&c.} = m \qquad (2),$$

and then substitute in the general formula these particular values.

If $n$ be not a positive integer, we must apply the same process to formula (II) instead of to formula (I).

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### Examples worked out.

1. Find the coefficient of $x^4$ in $(1 - x + x^2)^{-12}$.

The general term in this case is expressed by

$$(-12)(-13)\ldots\ldots(p+1)\,.\,1^p\,.\,\frac{(-x)^q}{\underline{|q}}\cdot\frac{(x^2)^r}{\underline{|r}},$$

and $p$, q, $r$ must satisfy the two following equations:—

$$\left.\begin{aligned} p + q + r &= -12 \\ q + 2r &= 4 \end{aligned}\right\}$$

| $p$ | $q$ | $r$ |
|---|---|---|
| − 14 | 0 | 2 |
| − 15 | 2 | 1 |
| − 16 | 4 | 0 |

The values will be found in the accompanying table.

The three terms corresponding to the three sets of values are

$$(-12)(-13)\cdot\frac{(x^2)^2}{\underline{|2}} + (-12)(-13)(-14\cdot\frac{(-x)^2}{\underline{|2}}\cdot\frac{x^2}{\underline{|1}}$$

$$+(-12)(-13)(-14)(-15)\cdot\frac{(-x)^4}{\underline{|4}}.$$

Which reduces to $(78 - 1092 + 1365)x^4$;

Or, $351x^4$.

2. Find the cube root of $(1 + x^{\frac{1}{2}} + x + x^{\frac{3}{2}})$ to the first four terms.

| $p$ | $q$ | $r$ | $s$ | |
|---|---|---|---|---|
| $-\frac{2}{3}$ | 1 | 0 | 0 | $x^{\frac{1}{2}}$ |
| $-\frac{5}{3}$ | 2 | 0 | 0 | $x$ |
| $-\frac{8}{3}$ | 3 | 0 | 0 | $\left.\begin{matrix}\\ \\ \end{matrix}\right\} x^{\frac{3}{2}}$ |
| $-\frac{2}{3}$ | 0 | 1 | 0 | |

The first term of the expansion is obviously 1. Using the values of $p, q, r, s$ corresponding to the first three powers of $x$, we have for the expansion,

$$1+\frac{1}{3}\cdot\frac{x^{\frac{1}{2}}}{\underline{|1}}+\frac{1}{3}\left(-\frac{2}{3}\right)\frac{(x^{\frac{1}{2}})^2}{\underline{|2}}+\frac{1}{3}\left(-\frac{2}{3}\right)$$

$$\left(-\frac{5}{3}\right)\cdot\frac{(x^{\frac{1}{2}})^3}{\underline{|3}}+\frac{1}{3}\cdot\frac{x^{\frac{3}{2}}}{\underline{|1}}.$$

Or, $$1+\frac{x^{\frac{1}{2}}}{3}-\frac{x}{9}+\frac{32x^{\frac{3}{2}}}{81}.$$

3. Find by the Multinomial Theorem the coefficient of $x^4$ in the expansion of $(1 - 2x + 3x^2 - 4x^3 +, \&c.)^{2n}$; and verify the result by the Binomial Theorem.

Here we need not take any terms beyond $5x^4$, so that we have to consider the expansion of

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4)^{2n}.$$

| $p$ | $q$ | $r$ | $s$ | $t$ |
|---|---|---|---|---|
| $2n-1$ | 0 | 0 | 0 | 1 |
| $2n-2$ | 1 | 0 | 1 | 0 |
| $2n-2$ | 0 | 2 | 0 | 0 |
| $2n-3$ | 2 | 1 | 0 | 0 |
| $2n-4$ | 4 | 0 | 0 | 0 |

The two equations to be satisfied are—

$$\begin{aligned} p+q+r+s+t &= 2n \\ q+2r+3s+4t &= 4, \end{aligned}$$

and the values are those given in the table. The whole coefficient will therefore be

$$\underline{|2n}\left\{\frac{1}{\underline{|2n-1}}\cdot 5+\frac{1}{\underline{|2a-2}}\cdot 2\cdot 3\right.$$

$$\left.+\frac{1}{\underline{|2n-1}}\cdot\frac{3^2}{1.2}+\frac{1}{\underline{|2n-3}}\cdot\frac{3^2}{1.2}\cdot\frac{3}{1}+\frac{1}{\underline{|2n-4}}\cdot\frac{2^4}{\underline{|4}}\right\}$$

which reduces to $\dfrac{4n\,(4n+1)\,(4n+2)\,(4n+3)}{1\,.\,2\,.\,3\,.\,4}$.

To verify this by the Binomial Theorem, we have

$$\begin{aligned}&(1-2x+3x^2-4x^3+,\ \&\text{c.})^{2n}\\ =\ &\{(1+x)^{-2}\}^{2n}\\ =\ &(1+x)^{-4n};\end{aligned}$$

and if the coefficient of $x^4$ in the Binomial expansion be found, it will be seen to agree with the result just obtained.

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## Examples LXXIII.

1. Find the term involving $x^2y^3$ in $(x+y+z)^5$; $abcde$ in $(a+b+c+d+e)^{10}$; of $x^2y^3z^{13}$ in $(x^2+xy+yz)^9$; $a^2bc$ in $(a-b+c)^4$; $a^2b^2c^2$ in $(a^2+b^2+c^2-ac-bc-ab)^3$; $x^{10}$ in $(1-x^2+x^4-x^6)^8$.

2. Find the coefficient of $x^{12}$ in $(1-2x+3x^2-4x^3+5x^4)^9$; of $x^r$ in $(1+x+x^2+,\ \&\text{c., } ad\ inf.)^n$; of the term which is independent of $x$ and $y$ in $\left(\dfrac{x^2}{y^2}-\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{y^2}{x^2}\right)^{20}$.

3. Show that if the coefficient of $a^2bc$ be known in the expansion of $(a+b+c)^n$, those of $ab^2c$ and $abc^2$ are also known.

4. Find the number of terms involving $x^4y$ in the expansion of $(x+x^{\frac{1}{3}}y^{\frac{2}{3}}+x^{\frac{2}{3}}y^{\frac{1}{3}}+x)^{15}$; and those involving $x^2$ in $\left(1-\dfrac{x^{\frac{1}{2}}}{2}+\dfrac{x}{3}-\dfrac{x^{\frac{2}{3}}}{4}+\dfrac{x^2}{5}\right)^{-12}$.

5. Find by the Multinomial and verify by the Binomial Theorem the following coefficients:—

Of $x^{2p}$ in $(1+2x+3x^2+,\ \&\text{c., } ad\ inf.)^{-1}$;

Of $x^n$ in $(1+3x+6x^2+10x^3+,\ \&\text{c., } ad\ inf.)^{-12}$;

Of $x^{\overline{p+3}}$ in $(1+nx+n^2x^2+,\ \&\text{c. } ad,\ inf.)^{\overline{p+3}}$;

Of $\dfrac{1}{x}$ in $(x^2+x+1+\dfrac{1}{x}+\dfrac{1}{x^2}+,\ \&\text{c., } ad\ inf.)^{-3}$.

6. Find the middle term of $(1-a+a^2-a^3)^{12}$, and the $p^{\text{th}}$ term in the expansion of $\left(\dfrac{x^2}{y^2}-\dfrac{x}{y}+1-\dfrac{y}{x}+\dfrac{y^2}{x^2}\right)^{2p}$.

7. Find to six terms the square root of $1+2x+5x^2-7x^3$; also to six terms the cube root of $(x^3-x+x^{-1}-x^{-3})$, in descending powers of $x$.

8. Expand as far as $x^5$,

$$\left(1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +, \text{\&c., } \textit{ad inf.}\right)\left(1 - \frac{x}{n}\right)^n$$

$$\left(\frac{1 + x + x^2}{1 - x + x^2}\right)^{-\frac{1}{2}}; \quad \frac{\left(x - \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} - \dfrac{x^7}{\underline{|7}}, \text{\&c.}\right)^{\frac{1}{2}}}{1 - \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}}, \text{\&c.}}.$$

# CHAPTER XVIII.

## INDETERMINATE COEFFICIENTS, PARTIAL FRACTIONS, AND RECURRING SERIES.

**175.** In Art. (168) it was proved that if the equality

$$A + Bx + Cx^2 +, \text{\&c}, = a + bx + cx^2 +, \text{\&c.}$$

hold for all values of $x$; then

$$A = a; \; B = b; \; C = c, \text{\&c.}$$

We shall now extend this proof to the case of two variables $x$ and $y$.

Let $A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y +, \text{\&c.}$
$= M + Nx + Py + Qx^2 + Rxy + Sy^2 + Tx^3 + Ux^2y +, \text{\&c.}$
for all values of $x$ and $y$, and where the variations of $x$ and $y$ are independent of each other,

Then shall $A = M; \; B = N; \; C = P, \text{\&c.}$

Let the equality be written in the form

$$\left.\begin{array}{l} A + Bx + Dx^2 + Gx^3 + \ldots \\ \;\; + Cy + Fy^2 + \ldots. \\ \;\; + Exy + Hx^2y + \ldots \end{array}\right\} = \left.\begin{array}{l} M + Nx + Qx^2 + Tx^3 + \ldots \\ \;\; + Py + Sy^2 + \ldots. \\ \;\; + Rxy + Ux^2y + \ldots. \end{array}\right\}$$

Since this is true for *all* values of $x$ and $y$, suppose $y$ to retain the value 0, while $x$ may receive any other values whatever. The equality becomes—

$$A + Bx + Dx^2 + Gx^3 + \ldots = M + Nx + Qx^2 + Tx^3 + \ldots$$

and this is true for *all* values of $x$.

Hence $A = M; \; B = N; \; D = Q; \; G = T$, &c. Art. (168).

Again, we may suppose $x$ to retain the value of 0, while $y$ receives any other values whatever.

We then have

$$Cy + Fy^2 + \ldots = Py + Sy^2 + \ldots \text{ for all values of } y.$$

Whence, as before,

$$C = P$$
$$F = S,$$
$$\text{\&c.} \quad \text{\&c.}$$

Again, in the third line, suppose $y$ to retain the value 1, while $x$ receives any other values whatever; we then have

$$Ex + Hx^2 + \ldots = Rx + Ux^2 +, \text{\&c.}$$

for all values of $x$; whence

$$E = R; \; H = U; \; \text{\&c.}$$

**176.** The application of this theory will be shown in the following examples.

1. Resolve the expression $\dfrac{1}{(x-1)(x-2)}$ into partial fractions.

Let $\dfrac{1}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$, where $A$ and $B$ are the quantities to be determined.

$$\therefore \quad \frac{1}{(x-1)(x-2)} = \frac{A(x-2) + B(x+1)}{(x-1)(x-2)}$$

Or,

$$1 = A(x-2) + B(x-1).$$

Since $x$ may receive any value whatever, let $x = 1$.

In this case, the above equation becomes

$$1 = -A$$

Or,

$$A = -1.$$

Next, let $x = 2$; then the equation becomes

$$1 = B.$$

Hence, by substitution for $A$ and $B$, we have

$$\frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1},$$

which are the partial fractions required.

2. Let $\dfrac{4}{(x+3)(x+2)(x+1)^3}$ be the expression to be resolved into partial fractions.

Let

$$\frac{4}{(x+3)(x+2)(x+1)^3} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+1)^3}$$
$$+ \frac{D}{(x+1)^2} + \frac{E}{x+1}.$$

Proceeding, as in the former examples, we have

$$4 = A(x+2)(x+1)^3 + B(x+3)(x+1)^3 + C(x+3)(x+2) +, \&c.$$

Let $x = -1$

Then $4 = 2C$

Or, $C = 2.$

The original equality may now be written

$$\frac{4}{(x+3)(x+2)(x+1)^3} - \frac{2}{(x+1)^3} = \frac{A}{x+3} +, \&c.$$

$$\text{Or,} \quad \frac{-2(x^2+5x+4)}{(x+3)(x+2)(x+1)^3} = \frac{A}{x+3} +, \&c.$$

Simplifying the first side of this, we have

$$\frac{-2(x+4)}{(x+3)(x+2)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{D}{(x+1)^2} + \frac{E}{x+1}.$$

Whence we get

$$-2(x+4) = A(x+2)(x+1)^2 + \ldots + D(x+3)(x+2) +, \&c.$$

Let $x = -1$

Then $-6 = 2D$

Or, $D = +3.$

Transposing $\frac{D}{(x+1)^2}$ to the first side of the equation, and substituting for $D$ its value, we get

$$\frac{-2(x+4}{(x+3)(x+2)(x+1)^2} + \frac{3}{(x+1)^2} = \frac{A}{x+3} + \frac{B}{(x+2)} + \frac{E}{x+1};$$

$$\text{Or,} \quad \frac{3x^2+13x+10}{(x+3)(x+2)(x+1)^2} = \frac{A}{x+3} +, \&c.,$$

which reduces to

$$\frac{3x+10}{(x+3)(x+2)(x+1)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{E}{x+1}.$$

The values of $A$, $B$, $E$ may now be found as in Ex. 1.

*Note.* Let the following two points be noticed in this example.

1°. When the Denominator contains a power of any factor, as $(x+1)^3$, a partial fraction must be assumed for that power, and for each lower power down to the first, $\frac{C}{(x+1)^3}$, $\frac{D}{(x+1)^2}$, $\frac{E}{x+1}$ in the example.

2°. When the Numerator corresponding to the highest power has been found, let that fraction be transposed to the first side of the equation, and be combined with the quantity already there; the expression on the first side will then reduce, as the Numerator and Denominator have a common factor; $(x+1)$ in the example. This process must be continued until only the first power of the said factor remains on the second side.

3. Resolve $\dfrac{3x^2+26x+83}{(x-1)(x^2+7x+20)}$ into partial fractions.

Let $$\frac{3x^2+26x+83}{(x-1)(x^2+7x+20)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+7x+20};$$

$$\therefore\ 3x^2+26x+83 = A(x^2+7x+20) + (Bx+C)(x-1).$$

Let $x = 1$; then, substituting for $x$ its value on both sides, we have $$112 = 28A$$

$$\therefore\ A = 4.$$

Next, let $x^2+7x+20 = 0$; or, $x^2 = -7x-20$.

Then we have

$$-21x-60+26x+83 = -7Bx-20B+Cx-Bx-C,$$

Or, $$5x+23 = (C-8B)x-20B-C \quad (a).$$

Hence, by the principle (see the note)

$$\left.\begin{aligned} 5 &= C-8B \\ 23 &= -C-20B \end{aligned}\right\}$$

From these two equations we find

$$B = -1$$
$$C = -3.$$

And the required partial fractions are

$$\frac{4}{x-1} - \frac{x+3}{x^2+7x+20}.$$

*Note.* When the Denominator of the given expression contains a quantity which cannot be resolved into simple factors, the Numerator corresponding to that quantity must be assumed to be a series of powers of $x$, the highest index of which is less by 1 than the highest index in the quantity itself. Thus, in example (3), the Numerator corresponding to the Denominator $x^2+7x+20$ was assumed to be $Bx+C$; for a Denominator $x^4+1$, the Numerator must be assumed to be $Bx^3+Cx^2+Dx+E$, and for $x^{2m}+1$ the Numerator must begin with $x^{2m-1}$.

Again, in the equality $(a)$ of example (3), it was assumed that if the equality hold for *two* values of $x$, the coefficients of like powers of $x$ on each side must be the same. And it may be assumed generally, that if an equality between two expressions of the $m^{\text{th}}$ degree in $x$ hold for $\overline{m+1}$ values of $x$, then the coefficients of like powers of $x$ on each side must be the same.

4. Extract the square root of $(1 + x + x^2)$.

Let $\sqrt{1 + x + x^2} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 +$, &c.

Squaring, we have

$$1 + x + x^2 = (1 + A^2x^2 + B^2x^4 +, \&c.) + 2(Ax + Bx^2 + Cx^3 + D^4 +, \&c.) + 2Ax(Bx^2 + Cx^3 +, \&c.) + 2Bx^2(Cx^3 + \ldots) + \text{higher powers of } x.$$

Collecting the coefficients of $x$, we have

$$1 + x + x^2 = 1 + 2Ax + (A^2 + 2B)x^2 + (2AB + 2C)x^3 + (B^2 + 2D + 2AC)x^4 +, \&c.$$

Hence we have

$$\begin{aligned} 2A &= 1, \text{ or } A = \tfrac{1}{2}; \\ A^2 + 2B &= 1, \text{ or } 2B = \tfrac{3}{4} \ \therefore\ B = \tfrac{3}{8}; \\ 2AB + 2C &= 0, \text{ or } 2C = -\tfrac{3}{8} \ \therefore\ C = -\tfrac{3}{16}; \\ 13 +^2 2D + 2AC &= 0, \text{ or } 2D = \tfrac{3}{64} \ \therefore\ D = \tfrac{3}{128}. \end{aligned}$$

And in the same way the coefficients may be found as far as we please.

$$\therefore\ \sqrt{1 + x + x^2} = 1 + \frac{x}{2} + \frac{3x^2}{8} - \frac{3x^3}{16} + \frac{3x^4}{128}, \&c.$$

5. To sum the series $1^4 + 2^4 + 3^4 + \ldots + n^4$.

Let $1^4 + 2^4 + \ldots + n^4 = A_o + A_1 n + A_2 n^2 + \ldots + A_p n^p$.

$$\therefore\ 1^4 + 2^4 + \ldots n^4 + (n+1)^4 = A_o + A_1(n+1) + A_2(n+1)^2 +, \&c. \ldots + A_p(n+1)^p.$$

By subtraction $(n+1)^4 = A_1 + 2A_2 n + A_2 + 3A_3 n^2 + 3A_8 n + A_3 + 4A_4 n^3 + 6A^4 n^2 + 4A_4 n + A_4 + 5A_5 n^4 + 10A_5 n^3 + 10A_5 n^2 + 5A_5 n + A_5 +, \&c. \ldots + pA_p n^{p-1}$.

Or, $n^4 + 4n^3 + 6n^2 + 4n + 1 = (A_1 + A_2 + A_3 +, \&c.) + (2A_2 + 3A_3 + 4A_4 + \ldots.)^n + (3A_3 + 6A_4 + 10A_5 + \ldots.)n^2 + (4A_4 + 10A_5 + \ldots)n^3$: &c. $+ \left((p-1)\mathrm{A}_{p-1} + \frac{p(p-1)}{1.2}A_p\right)n^{p-2} + pA_p n^{p-1}$.

Equating coefficients, we find

$$\begin{aligned} p.A_p &= 0, \ \therefore\ A_p = 0 \\ (p-1)A_{p-1} + + \frac{p(p-1)}{1.2}A_p &= 0, \ \therefore\ A_{p-1} = 0, \end{aligned}$$

and by continuing this process, it will appear that all the coefficients equal zero, until we come to the coefficient of $n^4$, which is $5A_5$.

$$\therefore\ 5A_5 = 1, \text{ or } A_5 = \tfrac{1}{5};$$

Also

$$\begin{aligned} 4A_4 + 10A_5 &= 4, \ \therefore\ A_4 = \tfrac{1}{2}; \\ 3A_3 + 6A_4 + 10A_5 &= 6, \ \therefore\ A_3 = \tfrac{1}{3}; \\ 2A_2 + 3A_3 + 4A_4 + 5A_5 &= 4, \ \therefore\ A_2 = 0; \\ A_1 + A_2 + A_3 + A_4 + A_5 &= 4, \ \therefore\ A_1 = -\tfrac{1}{30}. \end{aligned}$$

To find $A_0$, suppose $n = 0$, the equality then becomes
$0 = A_0$, which gives the value of $A_0$.

Hence, by substituting for $A_0$, $A_1$, &c., their values, we have

$$1^4 + 2^4 + 3^4 + \ldots + n^4 = -\frac{n}{30} + \frac{n^3}{3} + \frac{n^4}{2} + \frac{n^5}{5}.$$

$$= \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1)$$

$$= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).$$

*Note.* In the same way, the sum of a series of any other powers might be found, and it will always be seen that the index of the highest power of $n$ in the result will be greater by 1 than the index of each term in the series.

6. Expand the quantity $\frac{1}{(x-1)(x-2)}$ in a series of ascending powers of $x$, and find the general term.

The expression must first be resolved into partial fractions; this was done in example (1), where it was found that

$$\frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1}.$$

Now $\frac{1}{x-2} = -(2-x)^{-1} = -\frac{1}{2}\left\{1 - \frac{x}{2}\right\}^{-1}$

$$= -\frac{1}{2}\left\{1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{n}{2}\right)^r\right\}$$

And $-\frac{1}{x-1} = \frac{1}{1-x} = (1-x)^{-1}$,

$$= 1 + x + x^2 + \ldots + x^r + \ldots$$

The general term, or the term involving $x^r$ is seen to be

$$\left(-\frac{1}{2^{r+1}} + 1\right)x^r = \frac{2^{r+1}-1}{2^{r+1}}x^r.$$

By giving $r$ the values 0, 1, 2, 3, &c., in succession, the series is seen to be

$$\frac{1}{2} + \frac{3}{4}x + \frac{7}{8}x^3 + \frac{15}{16}x^4 +, \text{\&c.}$$

7. Sum the following series by resolving each term into partial fractions.

$$\frac{x}{(1-x)(1-ax)} + \frac{ax}{(1-ax)(1-a^2x)} +, \text{\&c., to } n \text{ terms.}$$

By the methods before used, we find

$$\frac{x}{(1-x)(1-ax)} = \frac{1}{1-a}\left(\frac{1}{1-x} - \frac{1}{1-ax}\right).$$

Similarly,

$$\frac{ax}{(1-ax)(1-a^2x)} = \frac{1}{1-a}\left(\frac{1}{1-ax} - \frac{1}{1-a^2x}\right).$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

$$\frac{a^{n-1}x}{(1-a^{n-1}x)(1-a^nx)} = \frac{1}{1-a}\left(\frac{1}{1-a^{n-1}x} - \frac{1}{1-a^nx}\right).$$

If we now add, cancel all like terms on the second side, and call the whole sum on the first side $S$, we find

$$S = \frac{1}{1-a}\left(\frac{1}{1-x} - \frac{1}{1-a^nx}\right).$$

The sum of this series, *ad inf.*, will be

$$\left.\begin{aligned} \frac{1}{1-a}\left(\frac{1}{1-x} - \frac{1}{1}\right) &= \frac{x}{(1-a)(1-x)} \quad \text{if } a < 1 \\ \frac{1}{1-a}\left(\frac{1}{1-x} - 0\right) &= \frac{1}{(1-a)(1-x)} \quad \text{if } a > 1 \\ &= \infty \quad \text{if } a = 1 \end{aligned}\right\}$$

8. Let $y = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} -$, &c., *ad inf.* (1.)

It is required to express $x$ in a series of powers of $y$.

Let $x = A_0 + A_1y + A_2y^2 + A_3y^3 +$, &c. . . . (2.)

If $-x$ be written for $x$, and $-y$ for $y$ in (1), it is seen that both sides of the equality change sign; now (2) must express the same relation between $x$ and $y$ as (1) does; therefore, when $-x$ is written for $x$ and $-y$ for $y$ in (2), both sides of the equality must change sign; and to secure this, all the powers of $y$ must be odd powers. Hence we may assume

$$x = A_1y + A_3y^3 + A_5y^5 +, \&\text{c.}$$

And
$$-\frac{x^3}{\underline{|3}} = -\frac{1}{6}y^3(A_1 + A_3y^2 + A_5y^4 + \ldots)^3.$$

$$= -\frac{1}{6}y^3(A_1^3 + 3A_1^2A_3y^2 +, \&\text{c.})$$

$$\frac{x^5}{\underline{|5}} = \frac{1}{120}y^5(A_1 +, \&\text{c.})^5$$

$$= \frac{A_1^5y^5}{120} + \text{higher powers of } y.$$

By Addition,

$$x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} = A_1y + y^3\left(A_3 - \frac{A_1^3}{6}\right) + y^5\left(A_5 - \frac{A_1^2A_3}{2} + \frac{A_1^5}{120}\right) +, \&\text{c.}$$

Or,
$$y = A_1 y + y^3\left(A_3 - \frac{A_1^3}{6}\right) + y^5\left(A_5 - \frac{A_1^2 A_3}{2} + \frac{A_1^5}{120}\right), \text{ \&c.}$$

Equating coefficients, we find

$$A_1 = 1$$

$$A_3 - \frac{A_1^3}{6} = 0.$$

$$\therefore \quad A_3 = \frac{1}{6}.$$

Similarly,
$$A_5 = \frac{A_1^2 A_3}{2} - \frac{A_1^5}{120}$$
$$= \frac{1}{12} - \frac{1}{120} = \frac{3}{40};$$

$$\therefore \quad x = y + \frac{y^3}{6} + \frac{3y^5}{40}, \text{ \&c.}$$

*Note.* The above is an example of "Reversion of Series."

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## Examples LXXIV.

1. Resolve the following expressions into partial fractions:—

$$\frac{x+37}{(x+3)(x+20)};\quad \frac{16}{a^2-64};\quad \frac{2x}{x^2-64};\quad \frac{40}{21x^2-58x+21};$$

$$\frac{x^2+3x}{x^3-6x^2+9x-4};\quad \frac{3x^2+3x+1}{(x+1)^3};\quad \frac{2+28x-x^2}{x(2-x)(1+x)(1-2x)^2};$$

$$\frac{x^2-x+1}{x^3(x^2-1)^2};\quad \frac{1+10x+9x^2}{(1+2x)^2(1+4x)^2};\quad \frac{4c^2x}{x^4+c^4};\quad \frac{4cx}{x^4+4c^4};$$

$$\frac{b^2+2ab}{a^3-b^3};\quad \frac{ab}{a^3-b^3};\quad \frac{15x^2}{x^4-11x^2+24};\quad \frac{6a^2b}{a^4+9a^2b^2+81b^4};$$

$$\frac{3(ab+ac+bc}{a^3+b^3+c^3-3abc};\quad \frac{1}{a^5+a^5}.$$

2. Find the partial fractions equivalent to $\dfrac{4x^4+2x^2+1}{x^2+3x+2}$.

*Note.* By actual division, this becomes $3x^2 - 9x + 23 - \dfrac{51x+45}{x^2+3x+2}$. The latter part can be resolved in the usual way. This method should always be used when the numerator is of dimensions equal to [illegible] higher than the denominator.

3. Resolve $\frac{1}{(x-a)(x-b)(x-c)}$ into simple fractions, the number of factors in the denominator being $n$; and hence show that the expression $\frac{1}{(a-b)(a-c)\ldots\ldots} + \frac{1}{(b-a)(b-c)\ldots\ldots} + \frac{1}{(c-a)(c-b)\ldots} +$, &c. $= 0$, if no two of the letters $a$, $b$, &c., be equal to one another.

4. Expand the following in a series of ascending powers of $x$:

$$\frac{2x+4}{x^2-2x+4}\,;\quad \frac{1-ax^2+bx^4}{1+ax^2+bx^4}\,;\quad \frac{3x-2}{(x-1)(x-2)(x-3)}\,;$$

and find the coefficient of $x^r$ in the last.

5. Find the value of the following to five terms:—

$$\sqrt[3]{1-2x+2x^2}\,;\quad \sqrt{1+\frac{n}{3}}\,;\quad \sqrt[5]{1+x^5}\,;\quad (1+2x+3x^2)^{-\frac{1}{2}}\,;$$

$$(1+4x+12x^2)^{-\frac{1}{2}}.$$

6. If $y^2 - xy - 1 = 0$, expand $y$ in terms of $x$ in two ascendidg series.

7. If $2y^3 - 3yx - 2 = 0$, express $y$ in a series of ascending powers of $x$.

8. From the following given relations between $x$ and $y$, expand $y$ in a series of ascending powers of $x$:—

$$y^3 - \frac{y}{x} + a = 0\,;\quad y^3 - 4ay(y-x) - 8x^3 = 0.$$

9. Express $x$ in terms of $y$ from the following relation:—

$$\frac{1}{3}(x^3-1) - xy = 0.$$

10. Reverse the following series:—

$$y = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3}, \text{ \&c. } \textit{ad inf.};$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}, \text{ \&c., } \textit{ad inf.};$$

$$y = a + bx + cx^2;\quad y = 1 + 2x + (2x)^2 + (2x)^3 +, \text{ \&c., } \textit{ad inf.}$$

$$x = 1 - \frac{y^2}{\underline{|2}} + \frac{y^4}{\underline{|4}} -, \text{ \&c., } \textit{ad inf.};$$

$$x = y + \frac{y^2}{\underline{|2}} + \frac{y^3}{\underline{|3}} +, \text{ \&c., } \textit{ad. inf.}$$

11. Sum the following series by the method of partial fractions:—

$$\frac{x}{(a+x)(a+bx)}+\frac{bx}{(a+bx)(a+b^2x}+\frac{b^2x}{(a+b^2x)(a+b^3x)}$$
+, &c., to *n* terms;

$$\frac{x}{(1-x)(1-abx)}=\frac{abx}{(1-abx)(1-a^2b^2x)}+,\text{ to } n \text{ terms.}$$

$$\frac{x}{x(x+1)}+\frac{x}{(x+1)(x+2)}+\frac{x}{(x+2)(x+3)},\ \&\text{c. to } n \text{ terms,}$$
and *ad inf.*

$$\frac{x}{(x+1)(x+4)}+\frac{x}{(x+2)(x+5)}+\frac{x}{(x+3)(x+6)},\ \&\text{c., to } n$$
terms, and *ad inf.*

12. Find the value of the following to *n* terms, and *ad inf.*:—

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+,\ \&\text{c.};\quad \frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+,\ \&\text{c.};$$

$$\frac{2}{3.6}+\frac{2}{4.7}+\frac{2}{5.8}+,\ \&\text{c.};\quad \frac{2}{1.3}+\frac{4}{3.7}+\frac{7}{8.15}+,\ \&\text{c.};$$

$$\frac{3}{2.5}+\frac{6}{5.11}+\frac{12}{11.23}+,\ \&\text{c.}$$

13. Sum, by reducing to partial fractions, the series given in Ex. (lxix.).

14. Resolve into partial fractions

$$\frac{1}{x^4+b^4+c^4-2b^2x^2-2c^2x^2-2b^2c^2}.$$

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## RECURRING SERIES.

**177.** It has been shown in the former part of this Chapter how fractional expressions can be resolved into partial fractions, and then be developed in a series of ascending powers of the variable $x$. The inverse problem is, when a series is given, to find the expression from which it is derived; the general term of the series can then be found, also the sum of $n$ terms.

Ex. 1. Find the expression of which the series $3-7x+20x^2-54x^3+148x^4$, &c., is the expansion.

Let $\dfrac{a+bx}{1+px+qx^2}$ be the expression required; $a, b, p, q$ being the indeterminate coefficients.

$$\therefore\ a + bx = (1 + px + qx^2)\,(3 - 7x + 20x^2 - 54x^3 +, \&c.$$

$$\text{Or, } a + bx = 3 + (3p - 7)\,x + (3q - 7p + 20)\,x^2 + (20q - 54p + 148)\,x^3, \&c.$$

Equating coefficients of like powers of $x$, we have

$$a = 3;\ \ b = 3p - 7;\ \ 3q - 7p + 20 = 0;\ \ 20q - 54p + 148 = 0.$$

Whence we find
$$\begin{aligned} a &= 3 \\ a &= -1 \\ p &= 2 \\ q &= 2. \end{aligned}$$

And the expression required is

$$\frac{3 - x}{1 + 2x - 2x^2}.$$

**178.** Series of this kind are called "Recurring Series," for a reason which will now be given.

Let the given series be $a_0 + a_1x + a_2x^2 +, \&c. + a_2x^2, \&c.,$ *ad inf.*

And let $\dfrac{a + bx}{1 + px + qx^2}$ be the expression from which the series is derived.

$$\begin{aligned} \therefore\ (a + bx) &= (1 + px + qx^2)\,(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ a_rx^r\ \ldots) \\ &= a_0 + (a_1 + pa)\,x + (a_2 + pa_1 + a_0)\,x^2 + \ldots\ldots \\ &\quad + (a_{r-1} + pa_{r-2} + qa_{r-3})\,x^{r-1} + (a_r + pa_{r-1} \\ &\quad + qa_{r-2})\,x^r +, \&c., \textit{ad inf.} \end{aligned}$$

Here it appears that the coefficients of any three consecutive terms after the second are connected by the invariable relation

$$a_r + pa_{r-1} + qa_{r-2} = 0.$$

By equating coefficients, as was done in Ex. (1), $a_0$ and $a_1$ may be found; then by the relation, $a_2$ may be found, then $a_3$, and so the coefficients of any number of terms may be found.

The denominator $1 + px + qx^2$ is called the "*Scale of Relation.*" It may consist of 2, 3, 4, or any greater number of terms.

Ex. 2. Find two more terms of the series $3 - 7x + 20x^2 - 54x^3 + 148x^4$, &c.

The scale of relation of this series was found in Ex. (1) to be $1 + 2x + 2x^2$. Hence the coefficients are connected by the relation $a_r + pa_{r-1} - 2a_{r-2} = 0$.

Let $r = 5\ \therefore\ a_5 + 2 \times 148 - 2 \times (-54) = 0\ \therefore\ a_5 = -404.$

$r = 6\qquad a_6 + 2\,(-404) - 2 \times 148 = 0\ \therefore\ a_6 = 1104.$

Ex. 3. Find the general term of the recurring series

$$1 + 2x + 2x^2 - 10x^3, \text{ \&c.}$$

By the method of Ex. (1), the expression from which the series is derived is found to be

$\dfrac{1 - 5x}{1 - 7x + 12x^2}$, which reduces to the partial fractions

$$\frac{2}{1 - 3x} - \frac{1}{1 - 4x}.$$

These expand into $2(1 + 3x + 3^2x^2 + \ldots + 3^r x^r + \ldots)$
$- (1 + 4x + 4^2x^2 + \ldots + 4^r x^r + \ldots)$.

And the general term is seen to be

$$(2 \cdot 3^r - 4^r)\, x^r.$$

Ex. 4. Find the sum of $n$ terms of the series $1 + 2x + 2x^2 - 10x^3$, &c.

The series has been shown in (2) to be equivalent to the two Geometric Progressions

$$2(1 + 3x + 3^2x^2 +, \text{ \&c. } ad\ inf.)$$
$$- (1 + 4x + 4^2x^2 +, \text{ \&c. } ad\ inf.)$$

And these, by the formula for the sum of a Geometric series, give for the whole sum

$$2 \cdot \frac{1 - (3x)^n}{1 - 3x} - \frac{1 - (4x)^n}{1 - 4x}.$$

*Note.* If the generating expression can be resolved into partial fractions, the general term and the sum of $n$ terms can be found as in this example. In other cases the process will be more difficult.

Ex. 5. Sum the recurring series $5 + 8 + 14 + 26 + 50$, &c., to $n$ terms.

If we take the series $5 + 8x + 14x^2 + 26x^3 + 50x^4$, &c., its generating expression will be found to be $\dfrac{5 - 7x}{1 - 3x + 2x^2}$, which resolves into the two partial fractions $\dfrac{2}{1 - x} + \dfrac{3}{1 - 2x}$; and these develop the two series

$$2(1 + x + x^2 + \ldots + x^{n-1} \ldots)$$
$$+ 3(1 + 2x + 4x^2 + \ldots 2^{n-1}x^{n-1} \ldots);$$

and if $x$ be put equal to 1 in each of these, the sum will be

$$2n + 3(2^n - 1),$$

Or, $$3 \cdot 2^n + 2n - 3,$$

which is the sum of $n$ terms of the above series.

Ex. 6. Find the generating expression of the series

$$x + 2x^2 + 5x^3 + 12x^4 + 27x^5, \text{ \&c.}$$

If we assume the expression to be $\dfrac{a + bx}{1 + px + qx^2}$, and proceed as in the former examples to equate the coefficients of $x$ on both sides, we shall find the values of $p$ and $q$ obtained from the different equations to be inconsistent with each other. Hence it must be concluded that the expression is not of the form assumed, but must consist of a greater number of terms. If we assume the expression to be $\dfrac{a + bx + ca^2}{1 + px + qx^2 + rx^3}$, and proceed as before, we shall find the expression to be $\dfrac{x^2}{(1 - 2x)(1 - x)^2}$.

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## Examples LXXV.

1. Find the expression from which the following series are derived, and write down the general term of the series :—

$4 + 9x + 21x^2 + 51x^3$, &c.; $5 + 13x + 35x^2 + 97x^3$, &c.;
$2 - x + 25x^2 - 37x^3$, &c.; $1 - 5x + 17x^2 - 53x^3$, &c.;
$3 - 2x + 4x^3 - 12x^4$, &c.; $2 + \dfrac{20}{3}x + \dfrac{104}{9}x^2 + \dfrac{656}{27}x^3$, &c.

2. Find the generating expression, also the sum of $n$ terms of each of the following :—

$4 + 2x + 2x^2 - 10x^3$, &c.; $5 + 8x + 14x^2 + 26x^3 + 50x^4$, &c.;
$7 + 19x + 1699x^2 + 6259x^3$, &c.; $4 - 12x + 64x^2 - 240x^3$
$+ 1024x^4 - 4032x^5$, &c.

3. Sum the following series to $n$ terms :—

$9 + 21 + 51 + 129$, &c.; $-8 + 4 + 148 + 1376$, &c.;
$1 + 3 + 11 + 43$, &c.; $-4 + 28 - 8 + 104 - 68$;
$1 + 11 + 89 + 659$, &c.; $+340 - 288 + 1360$, &c.
$1 - 11 + 97 + 779$, &c.

4. Find the expression from which the series $1 - x^2 + x^3 + 0$ is derived; also, that whose $n^{\text{th}}$ term is $\left\{3 \cdot 2^{n-1} - \left(-\dfrac{2}{3}\right)^{n-1}\right\}x^{n-1}$.

5. The scale of relation is $1 + ax + bx^2$, and the 5th and 6th terms are $(2a^5 - 3a^3b - 4ab^2)x^4$ and $-(2a^6 - 5a^4b - 3a^2b + 3b^6)x^5$, respectively : find the numerator of the generating expression.

6. The scale of relation is $1 - \overline{a + b}.x + abx^2$, and the general term of the series is $a^rb^r)x^{r-1}$: find the generating expression of the series.

# CHAPTER XIX.

## LOGARITHMS.

**179.** If some positive quantity different from unity be taken as a base, every number from 0 to $\infty$ may be regarded as a power of that base.

The "*Logarithm*" of a number is the index of the power to which the base must be raised to produce that number.

If 4 be the base, then
$$\left.\begin{aligned} 4^2 &= 16 \\ 4^{\frac{3}{2}} &= 8 \end{aligned}\right\}$$

And these equalities may be written
$$\left.\begin{aligned} \log.\ 16 &= 2 \\ \log.\ 8 &= \tfrac{3}{2} \end{aligned}\right\}$$

Similarly, if 10 be the base, then we have
$$10^2 = 100, \text{ or } \log._{10} 100 = 2.$$

Since $\dfrac{a^m}{a^n} = a^{m-n}$ always; if we put $m = n$;

Then $1 = a^0$, or $\log._a 1 = 0$ is true for any base.

**180.** One of the principal uses of logarithms is to shorten the processes of multiplication, division, involution, and evolution, of large numbers. For this purpose 10 is taken as the base, and the logarithms of all numbers are calculated and arranged in tables. The manner of constructing these is beyond the scope of the present work, but the following fundamental propositions are all that are necessary to enable the student to apply the tables to the calculation of products, quotients, powers, and roots.

i. *The logarithm of a quantity, which is the product of any number of factors, equals the sum of the logarithms of the several factors*; or
$$\log._a (mnp\ .\ .\ .) = \log._a m + \log._a n + \log._a p\ .\ .\ .\ .$$

Let $a$ be the base, $x$, $y$, $z$, the logs. of $m$, $n$, $p$, respectively; so that
$$\left.\begin{aligned} x &= \log._a m \\ y &= \log._a n \\ z &= \log._a p \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} m &= a^x \\ n &= a^y \\ p &= a^z \end{aligned}\right\}$$

Hence by multiplication, $mnp = a^{\overline{x+y+z}}$;

Or, $\log._a mnp = x + y + z$.

That is, $\log._a mnp = \log._a m + \log.\ n + \log._a p$.

The same proof may be extended to any number of factors.

ii. *The logarithm of the quotient of two numbers equals the difference between the logarithm of the dividend and the logarithm of the divisor.*

Or, $\log._a \left(\frac{m}{n}\right) = \log._a m - \log._a n$.

As before, $\left.\begin{aligned} m &= a^x \\ n &= a^y \end{aligned}\right\}$

By Division, $\frac{m}{n} = \frac{a^x}{a^y}$;

Or, $\frac{m}{n} = a^{x-y}$.

$\therefore$ $\log._a \frac{m}{n} = x - y$;

That is, $\log._a \frac{m}{n} = \log._a m - \log._a n$.

*Cor.* Hence it appears that the logarithm of any fractional expression equals the sum of the logarithms of the factors in the numerator, diminished by the sum of the logarithms of the factors in the denominator.

$$\text{Or, } \log.\left(\frac{m_1 m_2 m_3 m_4 \ldots}{n_1 n_2 n_3 n_4 \ldots}\right) = (\log._a m_1 + \log._a m_2 + \log._a m_3, \&c.) - (\log.\ n_1 + \log.\ n_2 + \log._a n_3, \&c.)$$

iii. *The logarithm of any power, whether integral or fractional, of a quantity, equals the logarithm of the quantity multiplied by the index of the power.*

Or, $\log._a (m\ ) = p.\log._a m$.

Let $\log._a m = x$, or, $a^x = m$.

Then $m^p = (a^x)^p$;

Or, $m^p = a^{px}$.

$\therefore$ $\log._a (m^p) = \log._a (a^{px})$.

$= px$;

Or, $\log._a (m^p) = p.\log._a m$.

*Note.* All roots are to be regarded as fractional powers.

When logarithms are tabulated for the purposes of calculation the base 10 is always selected; and these logarithms are said to form the "common system." The reasons for taking 10 as a base are stated in Art. 182.

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### Examples worked out.

**181.** The following examples are intended to show the use of the Tables, and to serve as specimens of the form in which logarithmic examples should be worked out. No explanations of the steps of the work are given, as these would spoil the model, but the reasons of the process should be carefully sought in i., ii., or iii. of Art. 180.

In the Tables here used the values of the decimal part, or *mantissa*, of the logarithms of all numbers, from 1 to 10,000, are tabulated to 7 places of decimals, and the characteristic is known by inspection. *See* Art. 182.

In applying the "differences" printed in the Tables the principle is assumed that a change in the number produces a proportional change in the value of its logarithm.

1. Having given a number, to find its logarithm.

Find log. 23.47529.

Log. 23.47500 = 1.3706056
Diff. for 20 = 37
" " 9 = 17
Log. 23.47529 = 1.3706110

2. Having given a logarithm, to find the number corresponding to it.

Let given log. = 3.2468980
Now log. 1765.6000 = 3.2468923
Diff. = 57
" for 200 = 49
80
" " 30 = 74
60
" " 6 = 48
∴ 1765.6236 = the number nearly.

*Note.* The "Mantissa" of a logarithm is always positive: the "Characteristic" may be either positive or negative; when it is negative, the minus sign is written *over* it.

Thus log. .004762 = $\bar{3}.6777894$;
Which means $-3 + (.6777894.)$

Again, let given log. = $\bar{7}.0986000$
Log. .00000012548000 = $\bar{7}.0985745$
Diff. = 255
„ for 700 = 242
130
„ „ 30 = 104
260
„ „ 8 = 276
∴ .00000012548738 = the number.

3. Find the value of $\frac{1.3048 \times .0025}{7.384}$ by logarithms.

Log. 1.3048 = 0.1155439
Log. .0025 = $\bar{3}.3979400$
$\bar{3}.5134839$
Log. 7.384 = 0.8682917
$\bar{4}.6451922$
Log. .00044176000 = $\bar{4}.6451864$
Diff. = 58
„ for 500 = 49
90
„ „ 90 = 88
20
„ „ 2 = 20
∴ .00044176592 = the result required.

4. Find the value of $\frac{(.015)^{\frac{1}{3}}}{(24.7658)^{\frac{5}{6}} + (.004)^{\frac{2}{9}}}$.

Log. $(.015)^{\frac{1}{3}} = \bar{3} + 1.1760913 \div 3$
$= \bar{1}.3920304$ (1)

$$
\begin{array}{llrcll}
\text{Log.} & & 24.7560 & = & 1.3936805 & \\
\text{Diff. for} & & 8 & = & 142 & \\
\hline
\text{Log.} & & (24.7568)^{\frac{5}{6}} & = & 1.3936947 \times \frac{5}{6} & \\
 & & & = & 6.9684735 \div 6 & \\
 & & & = & 1.1614123 & (2) \\
\text{Log.} & & (.004)^{\frac{2}{9}} & = & \bar{3}.6020600 \times \frac{2}{9} & \\
 & & & = & \bar{9} + 4.2041200 \div 9 & \\
 & & & = & \bar{1}.4671244 & (3) \\
\cline{5-5}
 & & & & 0.6285367 & (4) \\
\cline{5-5}
 & & & & \bar{2}.7634937 & (5) \\
\text{Log.} & & .058008000 & = & \bar{2}.7634879 & \\
\cline{5-5}
\text{Diff.} & & & = & 58 & \\
,, & \text{for} & 700 & = & 53 & \\
\cline{5-5}
 & & & & 50 & \\
 & & 60 & = & 45 & \\
\cline{5-5}
 & & & & 50 & \\
 & & 6 & = & 45 & \\
\hline
\therefore & & .058008766 & = & \text{the number required.} &
\end{array}
$$

*Note.* In the above example, (4) is the sum of (2) and (3); (5) is the difference of (4) and (1).

### Advantages of the Base 10.

**182.** In general the logarithm of a number will be a quantity made up of a whole number and a decimal fraction; the whole number is called the "*Characteristic*" of the logarithm, and the decimal is called the "*Mantissa.*" The first advantage of the base 10 is, that

*The characteristic is known by simple inspection of the number whose logarithm is required.*

For

$$
\begin{aligned}
10^0 &= 1 \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1000
\end{aligned}
$$

That is, if the *number* lie between 1 and 10, the characteristic = 0
,, ,, 10 and 100, ,, ,, = 1
,, ,, 100 and 1000, ,, ,, = 2
Generally, if number ,, $10^n$ and $10^{n+1}$ ,, ,, = $n$
when the number is greater than 1.

Again,

$$\begin{aligned} 10^0 &= 1 \\ 10^{-1} &= \cdot 1 \\ 10^{-2} &= \cdot 01 \\ 10^{-3} &= \cdot 001 \\ 10^{-4} &= \cdot 0001 \end{aligned}$$

That is, if the number lie between 1 and ·1, the characteristic = $-1$
,, ,, ·1 and ·01 ,, = $-2$
,, ,, ·01 and ·001 ,, = $-3$
Generally, if number ,, $10^{-n}$ and $10^{\overline{-n+1}}$ ,, = $-\overline{n+1}$
when the number is less than 1.

Hence in tables of common logarithms, the characteristics are not printed, but are written down from inspection.

2. The second advantage of the base 10 is, that

*If a number be multiplied or divided by any power of* 10, *the logarithm of the product or quotient has the same mantissa as the logarithm of the number itself has.*

Let $\log._{10}N = x$

$\therefore$ $10^x = N$

And $10^{x+2} = N \cdot 10^2$

$\therefore$ $x+2 = \log. (N \cdot 10^2)$

$$\left.\begin{aligned} \text{Or} \quad \log. (N.10^2) &= \log. N + 2 \\ \text{Similarly} \quad \log. \left(\frac{N}{10^2}\right) &= \log. N - 2 \end{aligned}\right\}$$

That is $\log. (N \cdot 10^2)$ and $\log. \left(\frac{N}{10^2}\right)$ differ only in their characteristics from log. N.

Thus

$$\begin{aligned} \log. 345 &= 2 \\ \log. 34500 &= 4 \\ \log. \cdot 000345 &= \bar{4} \end{aligned}$$

---

5. Given $\log. 2 = .3010300$
$\log. 7 = .8450980$

# CHAPTER XIX.

## LOGARITHMS.

**179.** If some positive quantity different from unity be taken as a base, every number from 0 to $\infty$ may be regarded as a power of that base.

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If 4 be the base, then $\left.\begin{aligned} 4^2 &= 16 \\ 4^{\frac{3}{2}} &= 8 \end{aligned}\right\}$

And these equalities may be written

$$\left.\begin{aligned} \log.\ 16 &= 2 \\ \log.\ 8 &= \tfrac{3}{2} \end{aligned}\right\}$$

Similarly, if 10 be the base, then we have

$$10^2 = 100, \text{ or } \log._{10} 100 = 2.$$

Since $\dfrac{a^m}{a^n} = a^{m-n}$ always; if we put $m = n$;

Then $1 = a^0$, or $\log._a 1 = 0$ is true for any base.

**180.** One of the principal uses of logarithms is to shorten the processes of multiplication, division, involution, and evolution, of large numbers. For this purpose 10 is taken as the base, and the logarithms of all numbers are calculated and arranged in tables. The manner of constructing these is beyond the scope of the present work, but the following fundamental propositions are all that are necessary to enable the student to apply the tables to the calculation of products, quotients, powers, and roots.

i. *The logarithm of a quantity, which is the product of any number of factors, equals the sum of the logarithms of the several factors*; or

$$\log._a (mnp\ .\ .\ .) = \log._a m + \log._a n + \log._a p\ .\ .\ .\ .$$

Let $a$ be the base, $x$, $y$, $z$, the logs. of $m$, $n$, $p$, respectively; so that

$$\left.\begin{aligned} x &= \log._a m \\ y &= \log._a n \\ z &= \log._a p \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} m &= a^x \\ n &= a^y \\ p &= a^z \end{aligned}\right\}$$

Hence by multiplication, $mnp = a^{\overline{x+y+z}}$;

Or, $\log_{.a} mnp = x + y + z$.

That is, $\log_{.a} mnp = \log_{.a} m + \log.\ n + \log_{.a} p$.

The same proof may be extended to any number of factors.

ii. *The logarithm of the quotient of two numbers equals the difference between the logarithm of the dividend and the logarithm of the divisor.*

Or, $\log_{.a}\left(\dfrac{m}{n}\right) = \log_{.a} m - \log_{.a} n$.

As before, $\left.\begin{aligned} m &= a^x \\ n &= a^y \end{aligned}\right\}$

By Division, $\dfrac{m}{n} = \dfrac{a^x}{a^y}$;

Or, $\dfrac{m}{n} = a^{x-y}$.

$\therefore$ $\log_{.a} \dfrac{m}{n} = x - y$;

That is, $\log_{.a} \dfrac{m}{n} = \log_{.a} m - \log_{.a} n$.

*Cor.* Hence it appears that the logarithm of any fractional expression equals the sum of the logarithms of the factors in the numerator, diminished by the sum of the logarithms of the factors in the denominator.

$$\text{Or, } \log.\left(\frac{m_1 m_2 m_3 m_4 \ . \ . \ .}{n_1 n_2 n_3 n_4 \ . \ . \ .}\right) = (\log_{.a} m_1 + \log_{.a} m_2 + \log_{.a} m_3, \text{ \&c.}) - (\log.\ n_1 + \log.\ n_2 + \log_{.a} n_3, \text{ \&c.})$$

iii. *The logarithm of any power, whether integral or fractional, of a quantity, equals the logarithm of the quantity multiplied by the index of the power.*

Or, $\log_{.a}(m\ ) = p.\log_{.a} m$.

Let $\log_{.a} m = x$, or, $a^x = m$.

Then $m^p = (a^x)^p$;

Or, $m^p = a^{px}$.

$\therefore$ $\log_{.a}(m^p) = \log_{.a}(a^{px})$.

$= px$;

Or, $\log_{.a}(m^p) = p.\log_{.a} m$.

*Note.* All roots are to be regarded as fractional powers.

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{2}{5}}}$$

Or,

$$x = 2 + \cfrac{1}{1 + \frac{5}{7}}$$

$$= 2\tfrac{7}{12}$$

$$\therefore \quad x = 2.58$$

This value of $x$ will be found correct to two places of decimals, and if the process be continued further, a still higher degree of accuracy may be obtained.

Another solution, by the aid of the Tables.

$$\text{Since} \quad 2^x = 6$$

$$\therefore \quad x.\log. 2 = \log. 6$$

$$\text{And} \quad x = \frac{\log. 6}{\log. 6}$$

$$= \frac{.7781513}{.3010300}$$

$$= 2.58, \text{ \&c. as before.}$$

## Examples LXXVI.

1. Find from the Tables the logs. of the following numbers :—

   23.4 ; 1.567 ; 200.08 ; .684927 ; .0046 ; .0198 ; .0000067892 4; .003928763 ; .1548279 ; $\frac{1}{4}$ ; $1.\dot{3}$ ; $.\dot{5}$ ; $144\frac{1}{16}$ ; $9\frac{2499}{2500}$ ; $10\frac{1}{10000000}$.

2. Find the numbers corresponding to the following logs. :—

   3.3162640 ; $\bar{1}.4444196$ ; $\bar{6}.5330134$ ; 1.2648701 ; 3.6008742 ; 0.5 ; $\frac{1}{3}$ ; $2\frac{1}{9}$ ; 1.5.

3. Find by logs. the values of the following products :—

   .19804 × 1418.025 ; .00017316 × 19.8651 ; $\dot{3}\dot{5} \times .0\dot{7}$ ; 746.2476 × .02096689 ; $94\frac{1}{5000} \times 2\frac{1}{25000}$ ; 2.4 × 56 × .0379 × 24.5 ; 061 × .47415 × 6.4314 × 9.394.

4. Find by logs. the following quotients :—

   104.8 ÷ 26.2 ; 568.54327 ÷ 7323 ; 4 ÷ 00255 ; 95642.37 ÷ .0879 ; .02097738 ÷ 23.5849 ; .0505 ÷ .0001953125.

5. Find by logs. the values of the following expressions :—

$$\frac{600 \times 25 \times 1618}{24 \times 5000.1}; \quad \frac{8352 \times 369}{(30.57)^3}; \quad \frac{.00376 \times (2.34)^4}{(15.26)^2};$$

$$\left(\frac{2133}{523.5}\right)^{12} \times \left(\frac{.03825}{9716}\right)^{10}.$$

6. Find by the Tables the following roots and powers :—

$$\sqrt{16.934994}; \quad \sqrt{.01595169}; \quad \sqrt{.00137641}; \quad \sqrt{.09765625};$$

$$\sqrt{.0000001215}; \quad \sqrt{116.316225}; \quad \sqrt[3]{1869.959168}; \quad \sqrt[3]{75.29};$$

$$\sqrt[3]{123.505992}; \quad \sqrt[3]{.005035382344}; \quad \sqrt[4]{600.3725};$$

$$\sqrt[7]{512}; \quad \sqrt[10]{.0874}; \quad \left(\frac{2}{3}\right)^{\frac{3}{2}}; \quad \left(\frac{3}{2}\right)^{\frac{2}{3}}; \quad \sqrt{.\dot{1}2628\dot{4} \times .00\dot{2}8571\dot{4}};$$

$$\sqrt[5]{\frac{17}{52}}; \quad \left(\frac{5}{4}\right)^{21}; \quad \left(\frac{3}{7}\right)^{\cdot 375}.$$

7. Find the values of the following :—

$$\frac{\sqrt[3]{24}}{\sqrt[4]{64}\,\sqrt[5]{5}}; \quad \left(\frac{3147}{27.64}\right)^{\frac{1}{4}}; \quad \left(\frac{.0275}{\sqrt{.0176}}\right)^{\frac{1}{3}}; \quad (.05916079)^{\frac{3}{7}}; \quad \frac{\sqrt[3]{24}}{\sqrt{.024}};$$

$$\frac{\sqrt[3]{0125}\,.\,\sqrt{31.45}}{.00081}; \quad 10^{\cdot 5}; \quad \frac{(5.20277)^3 \times (2138.07)^3 \times 8.15}{(103982}.$$

8. Find a third proportional to 10.75 and $.0\dot{6}$; to $(.25)^{\frac{1}{3}}$ and $(.75)^{\frac{1}{4}}$; to $(1.0005)^{\frac{2}{3}}$ and $(1.005)^{\frac{5}{2}}$.

9. Find a mean proportional to $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$; to $(10.75)^{\frac{2}{7}}$ and $(.075)^{\frac{2}{9}}$.

10. Find a fourth proportional to 10001, 10002, and 10003; to $3^9$, $2^6$, $5^4$; to 0.225, 0.4, and 0.1835; to 0.9, 1.6, and 0.734; to $\sqrt{1.027}$, $\sqrt[3]{2.546}$, and $(31.027)^2$.

11. Sum the following series by the aid of logarithms :—

9 + 36 + 144, &c., to 12 terms; 100 + 153 + 225, &c., to 20 terms.

12. Given

$$\left.\begin{aligned} \log. 2 &= .3010300 \\ \log. 3 &= .4771213 \\ \log. 7 &= .8450980 \end{aligned}\right\} \quad \text{Find } \log. 3600, \quad \log. 60, \quad \log. .03, \quad \log. 1.05, \quad \log. \sqrt{\frac{4}{343}}.$$

13. Given log. 1752 = 3.2435341

„ log. 1752.1 = 3.2435589,

Construct a table of proportional parts, and find log. 17.52087.

14. Given log. 1.3107 = .1175034
 " log. 13.108 = 1.1175364
 " log. 5 = .6989700,
Find the value of $\sqrt[17]{131072}$.

15. Given log. 125 = 2.096910, find log. 128 without the Tables.

16. What are the numbers whose logs. = 8, $\frac{1}{2}$, $-2$, $\frac{4}{3}$, $-\frac{1}{2}$, respectively?

17. If $x = \frac{(9.035)^5 - 1}{(9.035)^5 + 1}$, find $x$ by the Tables.

18. Find $x$ in each of the following equations :—

$5^x = 6$; $2^x = 19$; $(1.03)^x = 3.421763$; $9^x.4^{1-x} = 2^{2x} \times 27$.

$\left.\begin{array}{l} 2^x = 3^y \\ 4^x = 6^{y+1} \end{array}\right\}$; $8^{x+1}.125^{1-x} = 2^{4x+7}.5^{x+1}$; $2^{2x} - 9.2^x + 8 = 0$.

19. Find, without the Tables,

$$\text{Log.}_5\,(125)^{\frac{1}{2}},\ \log._4 8,\ \log._{10}.00001,\ \log._{10}\left(\frac{1}{\sqrt{2}}\right).$$

## CHAPTER XX.

### INTEREST, ANNUITIES, &c.

**183.** *Interest* is the sum paid for the use of money. The money lent is called the *Principal.*

In estimating the Interest due on any Principal, it is necessary to know the interest on a given sum for a given time. When the interest on £100 for one year is given, that quantity is called the *Rate per Cent.*

Thus, if the interest of £100 for one year be £3 10s., the rate per cent. is $3\frac{1}{2}$.

When the Principal remains constant, it is said to be at *Simple Interest ;* but when, at the end of every year or other fixed period of time, the Principal is increased by the Interest accumulated during that period, it is said to be at *Compound Interest.*

When the Principal and Interest for any time are added together, the sum is called the *Amount* for that time.

*Discount* is an allowance made for paying money before it becomes due.

**184.** To find the Simple Interest of a sum of money for any number of years.

Let $P$ = the principal in pounds.
$r$ = the interest of £1 for one year in pounds.
$n$ = the number of years.

Since interest for £1 for 1 year is £$r$
∴ interest for £$P$ „ 1 year is $Pr$
And interest of $P$ „ $n$ years is $Pnr$. (i.)

If $M$ denote the amount,

$$M = P + Pnr$$

Or, $$M = P(1 + nr). \qquad \text{(ii.)}$$

**185.** To find the Compound Interest of a sum of money for any number of years.

Take the same notation as in (184).

Then $P$ = the first principal.
$P(1 + r)$ = amount at end of first year.
Or, $P(1 + r)$ = the second principal.
∴ $\overline{P(1 + r)}\,.\,(1 + r)$ = amount at end of second year.
Or, $P(1 + r)^2$ = the third principal.
∴ $\overline{P(1 + r)^2}\,.\,(1 + r)$ = amount at end of third year.
Or, $P(1 + r)^3$ = amount at end of third year.

And, by continuing this process, it will appear that

$P(1 + r)^n$ = amount at end of $n$th year.

$$\therefore \quad M = P(1 + r)^n. \qquad \text{(iii.)}$$

$$\text{And interest} = P(1 + r)^n - P. \qquad \text{(iv.)}$$

**186.** To find the interest and amount of a sum of £$P$, for $n$ years, when the interest is paid $q$ times a year; each payment for £1 being $\frac{r}{q}$, at compound interest.

Here $P$ = the 1st principal;
$P\left(1 + \frac{r}{q}\right)$ = the amount at end of 1st period;

$$P\left(1+\frac{r}{q}\right)^2 = \text{the amount at end of 2nd period;}$$

&c. &c. &c.

$$\therefore \quad P\left(1+\frac{r}{q}\right)^{nq} = \text{the amount at end of } \overline{nq|}^{th} \text{ period, or at the end of } n \text{ years.}$$

$$\text{And} \quad P\left(1+\frac{r}{q}\right)^{nq} - P = \text{the interest at end of } n \text{ years.} \qquad \text{(v.)}$$

**187.** To find the present worth and discount of a sum due at the end of any time.

Let $D$ = the discount.
„ $P$ = the present worth.
„ $M$ = the sum due at the end of $n$ years.
„ $r$ = interest of £1 for 1 year.

Therefore at simple interest,

$$P(1+nr) = M;$$

$$\text{Or,} \quad P = \frac{M}{1+nr}. \qquad \text{(vi.)}$$

Also, from the nature of discount,

$$D = M - P,$$
$$= \frac{Mnr}{1+nr}. \qquad \text{(vii.)}$$

At compound interest,

$$P(1+r)^n = M,$$

$$\text{Or,} \quad P = \frac{M}{(1+r)^n}. \qquad \text{(viii.)}$$

$$D = M - P.$$
$$= M(1 - \overline{1+r|}^{-n}.) \qquad \text{(ix.)}$$

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## ANNUITIES.

**188.** To find the amount of an Annuity left unpaid for any number of years.

Let $A$ = the annuity in pounds.
„ $n$ = the number of years it is left unpaid.
„ $r$ = interest of £1 for 1 year.

Now £$A$ becomes due at the end of the first year.

This sum therefore bears interest for $\overline{n-1}$ years

Also £$A$ becomes due at the end of the second year.

This sum therefore bears interest for $\overline{n-2}$ years.

&c. &c. &c.

Hence, the amount of the first payment $= A(1+r)^{n-1}$;

" " second " $= A(1+r)^{n-2}$;

" " $\overline{n-1}|^{th}$ " $= A(1+r)$;

" " $n^{th}$ " $= A$.

The whole amount therefore equals the sum of the above series, or

$$\text{Amount of Annuity} = A.\frac{\overline{1+r}|^{n}-1}{r}. \qquad \text{(x.)}$$

**189.** To find the present value of an Annuity left unpaid for $n$ years.

In the last Article the amount for $n$ years was found to be

$$A.\frac{(1+r)^{n}-1}{r}.$$

And the present value of this must be, Art. (187),

$$A.\frac{(1+r)^{n}-1}{r}\times\frac{1}{(1+r)^{n}},$$

Or,

$$A.\frac{1-\overline{1+r}|^{-n}}{r}.$$

**190.** To find the present value of an Annuity to begin at the end of $p$ years and to continue $q$ years.

The value required is the present value of the Annuity for $(p+q)$ years, diminished by its present value for $p$ years, or, adopting the notation of the previous Articles,

$$A.\frac{1-\overline{1+r}|^{-\overline{p+q}}}{r}-A.\frac{1-\overline{1+r}|^{-p}}{r},$$

Or,

$$A.\frac{\overline{1+r}|^{-p}-\overline{1+r}|^{-\overline{p+q}}}{r}. \qquad \text{(xi.)}$$

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### Examples worked out.

1. Find the interest and amount of £400 for 3 years at 4 per cent., Simple Interest.

Generally, Interest $= Pnr.$ by (i.)

In this case, Interest $= 400 \times 3 \times \frac{4}{100}$

$= £48$

And Amount $= 400 + 48$

$= £448.$

2. Find the interest and amount of £400 for 3 years at 4 per cent., Compound Interest.

Generally, Amount $= P(1+r)^n.$ by (ii.)

In this case, Amount $= 400\,(1.04)^3$

$= £449$ 18*s.* 11*d.* nearly.

And Interest $= £49$ 18*s.* 11*d.*

*Note.* The difference in the results of (1) and (2) shows the advantage of Compound Interest over Simple.

3. At what rate per cent. will a principal of £*a* double itself in *n* years, at Compound Interest?

Generally, Amount $= P(1+r)^n.$

In this case, $2a = a(1+r)^n,$

Or, $(1+r)^n = 2.$

Whence, $n.\log.(1+r) = \log. 2;$

$\therefore \quad \log.(1+r) = \frac{\log. 2}{n}.$

Whence $(1+r)$ can be found from the Tables of Logarithms, and thus $r$ becomes known.

Then, rate per cent $= 100r.$

4. Find at what rate per cent. a sum will double itself at compound interest in 14 years.

By the last question

$$\log.(1+r) = \frac{\log. 2}{14}$$

$$= .0215021$$

By the Tables $1+r = 1.05076$

$\therefore \quad r = .05076$

And 5.076 is the rate per cent.

5. Find the interest on £250 put out to interest at 5 per cent. per annum for 10 years, compound interest, the payments being made half-yearly.

In this case the payment for every £100 at each half year is $\frac{5}{2}$, and for every pound .025;

Hence the amount $= 250\,(1.025)^{20}$.

*Note.* It is not strictly correct to call the interest in the last question 5 per cent. per annum. For it is evident that the amount of £100 for 1 year is $100\,(1.25)^2$

Or $105.0625$

The true rate per cent. per annum is therefore

$$5.0625 \text{ or } 5\tfrac{1}{16}.$$

6. If £$p$ be borrowed at $r'$ per cent. compound interest, and £$q$ be repaid every year, in how many years will the debt be paid off?

Let $n$ denote the number of years required;

Then the present value of the annuity of £$q$ for $n$ years must be £$p$. Art. (189).

If we put $$\frac{r'}{100} = r$$

Then $$q\,\frac{(1+r)^n - 1}{r\,(1+r)^n} = p$$

$$\therefore \quad (1+r)^n = \frac{q}{q - pr}$$

And $$n \log. (1+r) = \log. \left(\frac{q}{q-pr}\right)$$

$$\therefore \quad n = \frac{\log. \left(\frac{q}{q-pr}\right)}{\log. (1+r)}$$

To interpret this expression for $n$, when $\frac{q}{p} = r$.

Then $$q - pr = 0$$

And $\frac{q}{q-pr}$ becomes infinite,

And $\therefore \log. \left(\frac{q}{q-pr}\right)$ becomes infinite.

And $n$ is indefinitely great.

This is also evident from another consideration, for when $q = pr$, $q$ is exactly the interest of $p$, and it must always be paid, the debt still remaining the same.

## Examples LXXVII.

1. Find the simple interest of £150 for 2½ years at 4½ per cent; also the amount of £1015 for 9 months at 3¾ per cent.

2. What sum placed out at simple interest at 4 per cent. per annum on the day of a child's birth will amount to £1000 on the day that he is of age?

3. Work the last question with compound interest instead of simple.

4. Find general expressions for the

*Principal* in terms of the interest, rate, and time;
*Principal* „ „ amount, rate, and time;
*Interest* „ „ amount, rate, and time;
*Rate* „ „ interest, principal, and time;
*Rate* „ „ amount, principal, and time;
*Time* „ „ interest, principal, and rate;
*Time* „ „ amount, principal, and rate;
*Time* „ „ amount, interest, and rate;

in the case of simple interest.

5. Find the same general expressions as in (4) in the case of compound interest.

6. Find the amount of £1200 in 7 years at 5 per cent. comp. int.
„ „ 135 10*s*. in 15 years at 4 per cent. „
„ „ 772 6*s*. 11*d*. in 20 years at 3¼ per cent. „

7. Find what sum will amount to £109 19*s*. 1¼*d*. in 4 years at 5 per cent.

8. Find at what rate per cent. a sum of money will double itself in 14 years at compound interest.

9. What sum put out to compound interest for 3 years at 5 per cent. will amount to £115 15*s*. 3*d*.?

10. At what rate of compound interest must a sum be placed out so as to amount to $t$ times itself in $n$ years?

11. At what rate per cent. will £118 5*s*. produce £36 1*s*. 4*d*. at compound interest in 6 years?

12. Find the interest of £350 for 5 years at 2½ per cent. compound interest, the payments being made four times a-year.

13. Find the amount of £1000 at interest for 21 years at 5 per cent.; 1° at simple interest; 2° at compound interest; 3° at compound interest payable half-yearly.

14. A person lent £1200 on condition of his being paid back £1600 at the end of three years. What interest did he receive for his money, reckoning compound interest?

15. A merchant borrows £$P$ at $r$ per cent. per annum, and by trading makes it produce $r'$ per cent. What sum will he have gained at the end of $a$ years?

16. From a capital of £10,000 yielding 5 per cent. compound interest,

£800 is annually taken away. What capital will be left at the end of 10 years?

17. If a sum of £$a$ be borrowed at $r$ per cent. and £$b$ be repaid at the end of each year, in how many years will the debt be discharged at compound interest? Interpret the result when $\frac{b}{a} = \frac{r}{100}$.

18. A person who owes a debt of £$a$ wishes to extinguish the debt by $b$ annual payments. Find amount of each payment, reckoning compound interest at 4 per cent. per annum.

19. The amount of a sum of money in 4 years when the payments are annual is the same as the amount of the same sum in 3 years when the payments are quarterly. Find the rate per cent.

20. If a sum of money at a given rate of compound interest amount to $m$ times its original value in $a$ years, and to $n$ times its original value in $b$ years, prove that $\frac{b}{a} = \log_m n$.

21. At what rate per cent. would the present value of a debt of £1350 payable in 5 years be the same as that of a debt of £1200 payable in 3 years?

22. A debt of £1000 is to be discharged by monthly payments of £50. If it were proposed to pay the whole of this sum at once, when should the payment be made?

23. If £10,000 be given for a perpetual annuity of £300 a-year, what is the rate of interest?

24. If in the last question the annuity continue for 30 years, what is the rate of interest?

25. A gentleman wishes to divide £10,000 between his two sons; to one he gives £5,000 down, and to the other a perpetual annuity. Find how much it ought to be, interest being reckoned at 4½ per cent.

26. At what rate per cent. will an annuity of 100 guineas amount to £1534 19*s*. 10¾*d*. in 12 years?

27. Find the amount of an annuity of £50 for 7 years at 5 per cent.

28. Find the amount of an annuity of £150 for 10 years, payable quarterly, at 5 per cent. per annum.

29. Find the present value of an annuity of £300 for 20 years at 4 per cent.

30. Find the present value of an annuity of 100 guineas, to begin in 10 years and to continue for ever, at 4 per cent.

31. Find the difference between the present value of an annuity of £$a$ to continue for $n$ years, and the same annuity to continue for ever, the rate per cent. being $r$.

32. The present value of an annuity of £50 for 10 years is the same as the present value of an annuity of £20 to continue for ever. Find the rate of interest.

33. An annuity of £$a$ is to commence at the end of $p$ years and con-

tinue for $q$ years. Find the equivalent annuity to commence at once and continue for ever.

34. An annuity of £200 is to begin in 14 years and continue for 7 years. Find the present worth of it; also the equivalent perpetual annuity to begin at once.

35. A gentleman wishes to secure a sum of £200 a-year for the education of his son; the payments to begin when he is 14 and to continue till he is 21. What annual sum paid for 14 years from the day of his birth will secure this?

36. What should be the purchase-money for a fee-simple of £500 a-year, the purchaser to take possession at the end of 10 years, so that he may make 5 per cent. of his money?

37. A property yielding £$a$ per annum is given by a testator, 1° to a charity for $m$ years; 2° to another charity for $n$ years, and after that to a third charity for ever. Find the value of each bequest.

38. Which is the better interest, 5 per cent. payable quarterly, or $5\frac{1}{8}$ per cent. payable yearly?

39. If a lease of an estate for 21 years be bought for £10,000, and the annual rental be £750, what rate per cent. does the purchaser receive for his money?

40. If 7 years of a lease of 21 years have lapsed, and the whole rental be £560, subject to a reserved rent of £25, what ought to be given for the renewal of the lease, interest being reckoned at 4 per cent?

## CHAPTER XXI.

### SUMMATION OF SERIES, PILES OF BALLS, SHELLS, &c.

**191.** In this Chapter some examples of the summation of series will be given, in which the series are of different forms from those which have been already treated of.

i. To sum the series

$$(a+b)(2a+b)(3a+b)+(2a+b)(3a+b)(4a+b)+,\ \&\text{c.},$$
to $n$ terms.

$$\text{Let}\quad \left.\begin{array}{l}(a+b)(2a+b)(3a+b)(4a+b)\\ +(2a+b)(3a+b)(4a+b)(5a+b)\\ +\qquad \&\text{c.}\qquad \&\text{c.}\qquad \&\text{c.}\\ +(na+b)(\overline{n+1}\,a+b)(\overline{n+2}\,a+b)(\overline{n+3}\,a+b)\end{array}\right\}=S.$$

Then we shall have

$$\left.\begin{array}{l} b(a+b)(2a+b)(3a+b) \\ +(a+b)(2a+b)(3a+b)(4a+b) \\ \qquad \&c. \qquad \&c. \qquad \&c. \\ +(\overline{n-1}\,a+b)(na+b)(\overline{n+1}\,a+b)(\overline{n+2}\,a+b) \end{array}\right\}$$

$$=\left\{\begin{array}{l} S+b(a+b)(2a+b)(3a+b) \\ -(na+b)(\overline{n+1}\,a+b)(\overline{n+2}\,a+b)(\overline{n+3}\,a+b). \end{array}\right.$$

By subtracting in order, the corresponding lines of the lower series from those of the upper, we get

$$\left.\begin{array}{l} 4a(a+b)(2a+b)\,3a+b \\ +4a(2a+b)(3a+b)(4a+b) \\ \qquad \&c. \qquad \&c. \qquad \&c. \\ +4a(na+b)(\overline{n+1}\,a+b)(\overline{n+2}\,a+b) \end{array}\right\}$$

$$=\left\{\begin{array}{l} (na+b)(\overline{n+1}\,a+b)(\overline{n+2}\,a+b)(\overline{n+3}\,a+b) \\ \qquad -b(a+b)(2a+b)(3a+b). \end{array}\right.$$

Hence, the sum required

$$=\frac{(na+b)(\overline{n+1}.a+b)(\overline{n+2}.a+b)(\overline{n+3}.a+b)-b(a+b)(2a+b)(3a+b)}{4a}.$$

ii. Sum the series

$$\frac{1}{(a+b)(a+2b)(a+3b)}+\frac{1}{(a+2b)(a+3b)(a+4b)}\cdots$$
$$+\frac{1}{(a+nb)(a+\overline{n+1}b)(a+\overline{n+2}b)}.$$

Let $\dfrac{1}{a(a+b)}+\dfrac{1}{(a+b)(a+2b)}+\ldots+\dfrac{1}{(a+\overline{n-1}b)(3+nb)}=S.$

Then

$$\frac{1}{(a+b)(a+2b)}+\frac{1}{(a+2b)(a+3b)}+\ldots+\frac{1}{(a+nb)(a+\overline{n+1}b)}$$

$$\left.\begin{array}{l} =s \\ -\dfrac{1}{a(a+b)} \\ +\dfrac{1}{(a+nb)(a+\overline{n+1}b)} \end{array}\right\}$$

By subtracting, in order, the terms of the lower series from the corresponding terms of the upper series, we get

$$\frac{2b}{a(a+b)(a+2b)}+\frac{2b}{(a+b)(a+2b)(a+3b)}+\cdots\cdots$$

$$+\frac{2b}{(a+\overline{n-1}b)(a+nb)(a+\overline{n+1}b)}=\frac{1}{a(a+b)}$$
$$-\frac{1}{(a+nb)(a+\overline{n+1}b)}.$$

Hence, the sum required

$$=\frac{1}{2b}\left(\frac{1}{a(a+)}-\frac{1}{(a+nb(a+\overline{n+1}b)}.\right.$$

This method may be used to sum a series of the above form, when each term consists of four or five, or any greater number of factors in the denominator.

The rule is to assume $S$ to be the sum of a series similar to the given series, and having one factor less in each of its terms than the terms of the given series have.

iii. To find the sum of the series

$\frac{1}{a}+\frac{c}{a(a+b)}+\frac{c(c+b)}{a(a+b)(a+2b)}+$, &c., to $n$ terms; and *ad infinitum*.

Let
$$\frac{c}{a}+\frac{c(c+b)}{a(a+b)}+\frac{c(c+b)(c+2b)}{a(a+b)(a+2b)}\;.\;.\;.\;.\;.\;.\;.\;.\;,$$
$$+\frac{c(c+b)\;.\;.\;.\;(c+\overline{n-1}b)}{a(a+b)\;.\;.\;.\;(a+\overline{n-1}b)}=S;$$

$$\therefore\;1+\frac{c}{a}+\frac{c(c+b)}{a(a+b)}+\ldots+\frac{c(c+b)\;.\;.\;.\;(c+\overline{n-2}b)}{a(a+b)\;.\;.\;.\;(a+\overline{n-2}b)}$$
$$=s+1-\frac{c(c+b)\;.\;.\;.\;(c+\overline{n-1}b)}{a(a+b)\;.\;.\;.\;(a+\overline{n-1}b)}.$$

By subtracting, as in the former cases, we find

$$\frac{c-a}{a}+\frac{c(c-a)}{a(a+b)}+\,.\,.+\frac{c(c+b)(c+2b)\,.\,.\,(c+\overline{n-2}b)(c-a)}{a(a+b)(a+2b)\,.\,.\,(a+n-2b)(a+\overline{n-1}b)}$$
$$=\frac{c(c+b)\;.\;.\;.\;(c+\overline{n-1}b)}{a(a+b)\;.\;.\;.\;(a+\overline{n-1}b)}-1.$$

Hence, sum required

$$=\frac{1}{c-a}\left\{\frac{c(c+b)\ldots(c+\overline{n-2}b)}{a(a+b)\ldots(a+\overline{n-2}b)(a+\overline{n-1}b)}-1\right\}\text{ to } n \text{ terms.}$$

The limit of the sum of the series, when continued *ad infinitum*, is

$$\frac{1}{a-c}.$$

iv. The series $\frac{c}{a(a+b)(a+2b)} + \frac{2c}{(a+b)(a+2b)(a+3b)} + \ldots,$

$+ \frac{3c}{(a+2b)(a+3b)(a+4b)} +,$ &c.

may be resolved into the three series

$$\frac{c}{b} \cdot \frac{1}{a(a+2b)} + \frac{c}{b} \cdot \frac{1}{(a+2b)(a+4b)} + \frac{c}{b}\,\frac{1}{(a+4b)(a+6b)}, \text{ \&c.}$$

$$+ \cdot \frac{c}{b}\,\frac{1}{(a+b)(a+3b)} + \frac{c}{b}\,\frac{1}{(a+3b)(a+5b)} + \text{\&c.}$$

$$-\left(\frac{ac}{b}\,\frac{1}{a(a+b)(a+2b)} + \frac{ac}{b} \cdot \frac{1}{(a+b)(a+2b)(a+3b)} + \text{\&c.}\right);$$

and each of these may be summed as shown in the previous examples.

The process used in the above cases is called the method of *Subtraction.*

The series given for Summation in Ex. (lxxviii.) will fall under one or other of the forms which have just been discussed.

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## POLYGONAL NUMBERS.

**192.** The sum of an Arithmetic Series, whose first term is 1, and com. diff. 1 $= \frac{n^2 + n}{2}$

„ 1, „ 2 $= n^2$

„ 1, „ 3 $= \frac{3n^2 - n}{2}$

„ 1, „ 4 $= \frac{4n^2 - 2n}{2}$

&c. &c. &c.

„ 1, „ $(m-2)$ $= \frac{mn^2 - \overline{m - 2n}}{2}$

If, in each of these formulæ, we give $n$ the values 1, 2, 3, &c., in succession, we shall obtain a series of numbers, the properties of which we proceed to discuss.

All numbers of the form $\frac{n^2 + n}{2}$, or 1, 3, 6, 10, 15, 21, &c., are called "*Polygonal Numbers*" of the 3rd order, or "*Triagonal*" Numbers, and sometimes "*Triangular*" Numbers.

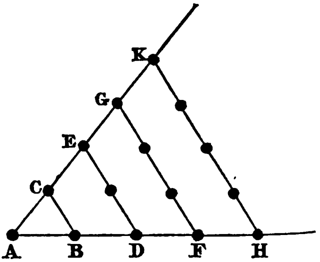
Those numbers of the form $n^2$, or 1, 4, 9, 16, 25, 36, &c., are called "*Polygonal numbers*" of the 4th order, or "*Quadragonal*" numbers.

All numbers of the form $\frac{3n^2 - n}{2}$, or 1, 5, 12, 22, 35, &c., are called "*Polygonal numbers*" of the 5th order, or "*Pentagonal*" numbers.

And generally numbers of the form $\frac{(m-2)n^2 - (m-4)n}{2}$ are called "*Polygonal*" *numbers of the $m$th order.*

The names Triagonal, Quadragonal, &c., may be geometrically illustrated as follows:—

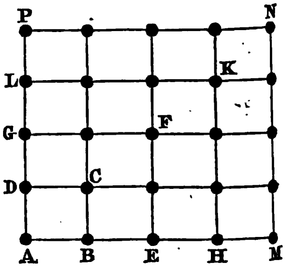
Let $ABC$, $ADE$, $AFG$, $AHK$, &c., be a series of equilateral triangles, whose sides are 1, 2, 3, 4, &c., units respectively; and let points be placed at equal distances of one unit along the sides, as in the figure



| Then | | | | |
|---|---|---|---|---|
| | $ABC$ | contains | 4 | points |
| | $ADE$ | ,, | 6 | ,, |
| | $AFG$ | ,, | 10 | ,, |
| | $AHK$ | ,, | 15 | ,, |
| | &c. | | &c. | |

And these numbers form the series of Triagonal numbers.

Next, if a series of squares be constructed whose sides are 1, 2, 3, &c., units in length, and points be placed as before, we have the series 1, 4, 9, 16, 25, &c., which form the series of Quadragonal numbers.



Similarly, if a series of pentagons be constructed, the number of points on the 1st, 2nd, 3rd ... figure will represent the pentagonal numbers, 1, 5, 12, 22, &c.

**194.** To find the sum of $n$ terms of a series of polygonal numbers of the $r$th order.

The general term is $\frac{(r-2)n^2 - (r-4)n}{2}$.

Hence, the sum of $n$ terms will be

$\frac{1}{2}(r-2)(1^2 + 2^2 + \ldots + n^2) - \frac{1}{2}(r-4)(1 + 2 + 3 + \ldots + n$

Or, $\frac{r-2}{2} \cdot \frac{n(n+1)(2n+1)}{6} - (r-4)\frac{n(n+1)}{4}$,

which becomes $\frac{n(n+1)}{6}\{n-1)(r-2)+3\}$.

---

## PILING OF BALLS AND SHELLS.

**194.** The heaps in which Cannon-Balls are piled are of ee different forms.

1°. *When all the layers are equilateral triangles, and the heap alls is a pyramid with an equilateral triangle for the base.*

Let $n$ be the number of balls in one side of the base.

eginning from the top, the numbers in the successive layers are , 6, 10, &c., which is the series of triagonal numbers to $n$ terms.

The general term of the series is

$$\frac{r^2+r}{2},$$

nd the sum of $n$ terms will therefore be

$$\tfrac{1}{2}\{1^2+2^2+\ldots+n^2\}+\tfrac{1}{2}\{1+2+\ldots+n\}$$

Or, $\frac{1}{12}n(n+1)(2n+1)+\frac{1}{4}n(n+1)$

Or, $\frac{1}{6}n(n+1)(n+2)$, (1.)

The required number in the pile.

°. *When all the layers are squares, and the heap of balls is a amid with a square base.*

Let $n$ be the number of balls in one side of the base.

The number of balls in each layer will be

1, 4, 9, 16, 25, &c., to $n$ terms.

Or, $1^2+2^2+3^2+\ldots+n^2$.

Which equals $\frac{1}{6}(n)(n+1)(2n+1)$. (2.)

°. *When each layer is a rectangle, but not a square.*

Let $n$ be the number of balls in the length of the base,

$m$ ,, ,, breadth ,,

Also, let $n-m=p$.

hen the top layer will be a single row of $\overline{p+1}$ balls.

And the sum of the numbers of balls in the different layers will be

$$1(p+1)+2(p+2)+3(p+3)+\ldots+m(p+m),$$

$$\text{Or,}\quad (1^2+2^2+\ldots+m^2)+p(1+2+3+\ldots+m),$$

$$\text{which give}\quad \tfrac{1}{6}m(m+1)(2m+1)+\frac{p}{2}m(m+1),$$

$$\text{Or,}\quad \tfrac{1}{6}m(m+1)(3p+2m+1). \qquad (3.)$$

Since $n-m=p$; the expression may be written

$$\tfrac{1}{6}m(m+1)(3n-m+1). \qquad (4.)$$

The first of these formulæ should be used when the number of balls in the upper row and the number of layers are given; the second when the number of balls in the length and breadth of the base is given.

When the number of balls in an incomplete pile of balls is required we must proceed as follows:—

*First, find the number of balls in the complete heap which has the lowest course or layer for its base; secondly, find the number of balls in the complete heap which has the top course for its base; thirdly, subtract the latter number from the former.*

### Examples LXXVIII.

1. Sum the following series to $n$ terms.

$5.8.11+8.11.14+11.14.17$, &c.; $4.6+6.8+8.10$, &c.; $\frac{7}{6}\times\frac{5}{3}+\frac{5}{3}\times\frac{13}{6}+\frac{13}{6}\times\frac{8}{3}$, &c.; $5.6.7.8+6.7.8.9+7.8.9.10$, &c.

2. Write down the $n^{\text{th}}$ term, and find the sum of $n$ terms of the following series:—

$$\frac{1}{7.13}+\frac{1}{13.19}+\frac{1}{19.25},\ \&c.;\quad \frac{1}{8.13.18}+\frac{1}{13.18.23},\ \&c.;$$

$$\frac{1}{4.6.8.10}+\frac{1}{6.8.10.12}+\frac{1}{8.10.12.14},\ \&c.$$

3. Find the sum of $n$ terms; also the limit of the sum of the following series, continued *ad inf.*:—

$$\frac{1}{1}+\frac{5}{1.2}+\frac{5.6}{1.2.3}+\frac{5.6.7}{1.2.3.4},\ \&c.$$

$$\frac{1}{2} + \frac{3}{2.7} + \frac{3.8}{2.7.12} + \frac{3.8.13}{2.7.12.17}, \text{ \&c.}$$

$$\frac{1}{4} + \frac{5}{4.7} + \frac{5.8}{4.7.12} + \frac{8.13.18}{4.1.10.13}, \text{ \&c.}$$

4. Find the sum to *n* terms, and *ad infinitum*, of the following series:—

$$\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{3.7.9}, \text{ \&c.}; \frac{1}{2.3.4} + \frac{2}{3.4.5}, \text{ \&c.};$$

$$\frac{1}{2.3.4.5} + \frac{2}{3.4.5.6} + \frac{3}{4.5.6.7}, \text{ \&c.};$$

$$\frac{3}{5.9} + \frac{6}{9.13} - \frac{9}{13.17}, \text{ \&c.}$$

5. Sum the following to *n* terms, and *ad infinitum*:—

$$\frac{1}{a^2} + \frac{c}{a^2(a^2+2b^2)} + \frac{c(c+2b^2)}{a^2(a^2+2b^2)(a^2+3b^2)}, \text{ \&c.}$$

$$(2a^2-b^2)(3a^2-b^2)+(3a^2-b^2)(4a^2-b^2)+(4a^2-b^2)(5a^2-b^2), \text{ \&c.}$$

$$\frac{1}{(a+5b)(a+6b)(a+7b)(a+8b)} + \frac{1}{(a+6b)(a+7b)(a+8b)(a+9b)}, \text{ \&c.}$$

---

## Examples LXXIX.

1. Find the number of balls in a triangular heap, the number of balls in one side of the base being 34.

2. Find the number of balls when the number in one side of the base is 50, also when it is 70.

3. Find the number of balls in the following complete triangular heaps:—

1°. When the number of layers is 12.
2°. „ „ 16.
3°. „ „ 18.

4. Find the number of balls in an imperfect triangular heap, there being 36 balls in one side of the base, and the number of layers being 16.

5. An imperfect triangular heap of balls has 50 balls in each side of the base, and 24 in each side of the top layer. Find the number of balls.

6. In an imperfect triangular heap of 6 layers the top layer contains 76 balls. Find the whole number of balls in the heap.

7. 1540 balls are to be made into a complete triangular heap. Find the number of balls in each side of the base.

8. 920 balls are to be piled in an incomplete triangular heap of 8 layers. Find the number of balls in each side of the base.

9. Two complete triangular heaps of balls contain 1495 balls, and the greater heap contains 1165 balls more than the other. Find the number of balls in the base of each.

10. The base of a triangular heap of balls contains 20 balls more than the next layer. Find the number of balls in the base.

11. How many balls are there in a complete square heap when the number in one side of the base is 48?

12. Find the number of balls in the following square heaps, which have in one side of the base 100, 55, and 23 balls respectively.

13. If a complete square heap consist of 30 layers, how many balls are there in the heap?

14. If the middle layer of a complete square heap contain 121 balls, find how many there are in the whole heap.

15. If the middle layer of a complete triangular heap contain 66 balls, how many rows are there?

16. Find the number of balls in an imperfect square heap of 10 layers, the number in the base being 1600.

17. An imperfect square heap of balls contains 10 layers, and the number of balls in the base is four times as great as the number in the top layer. Find the whole number of balls.

18. How many balls of 6 inches diameter can be piled on a space of ground 10 yards square, the heap not being more than 6 feet high.

19. How many balls are there in the base of a square heap which contains 2870 balls.

20. The number of balls in a complete square heap : the number of balls in a complete triangular heap : : 7 : 4. Find the number of balls in each heap.

21. Find the number of balls in a rectangular pile which has 24 balls in the length, and 16 balls in the breadth of the base.

22. Find the number of balls in the two following heaps: 1°, when the base has 105 in the length and 20 in the breadth; 2°, when it has 40 in the length and 30 in the breadth.

23. The top row of a rectangular pile has 20 balls, and there are 10 courses. Find the number of balls in the pile.

24. Find the difference between the number of balls in a square pile of 50 courses, and two complete rectangular piles each having 50 in the length and 25 in the breadth of the base.

25. Find the number of balls in an incomplete rectangular heap, whose base contains 40 × 30 balls, and which has 10 courses.

26. If the base of an incomplete rectangular pile of 10 courses contain 1911 balls, and the upper course contain 1131 balls, find the number of balls in the heap.

27. Compare the number of balls in a complete square pile with the number in a complete triangular pile which has the same number in one side of its base as there are in one side of the square base, and find the numbers in one side of the base of each, of which the ratio is 1·88.

*28. If n* be the whole number of balls in a complete square pile, show that the number of courses in the pile is the integer next below the cube root of $3n$.

29. If the pile be triangular, show that the number of courses is the integer next below $\sqrt[3]{6n}$.

# APPENDIX.

## APPLICATION OF THE METHOD OF FINDING THE SQUARE ROOT TO NUMBERS.

To adapt the method of Art. (43) to numbers, it is necessary to observe that every number may be put in the form of a series of powers of 10.

$$\begin{array}{lrl}
\text{Thus} & 5184 & = 5.13^3 + 1.10^2 + 8.10 + 4 \\
 & & = 51.10^2 + 8.10 + 4 \\
 & & = 49.10^2 + 28.10 + 4 \\
\text{which is of the form} & & \phantom{=} 49x^2 + 28x + 4
\end{array}$$

$\therefore$ 7.10 + 2 or 72 is the square root of 5184.

It would however be too laborious to begin by putting large numbers into the form of a perfect algebraical square; instead of this, we may proceed as in the subjoined example.

To find the square root of 75625.

$$\begin{array}{rl}
 & 7.10^4+5.10^3+6.10^2+2.10+5\ (2.10^2+7.10+5 \\
 & 4.10^4 \\ \hline
 & 3.10^4+5.10^3+6.10^2 \\
4.10^2+7.10\,) = & 35.10^3+6.10^2+ \\
 & 28.10^3+49.10^2 \\ \hline
 & 7.10^3-43.10^2+2.10+5 \\
\left.\begin{array}{r} 4.10^2+14.10+ \\ = 5.10^3+4.10+5 \end{array}\right) = & 27.10^2+2.10+5 \\
 & 25.10^2+20.10+25 \\ \hline
 & 2.10^2-18.10-20 = 0 \\ \hline
\end{array}$$

And the process may be abridged as follows:

$$\begin{array}{rr}
 & \dot{7}5\dot{6}2\dot{5}\ (275 \\
 & 4 \\ \hline
47\,) & 356 \\
 & 329 \\ \hline
545\,) & .2725 \\
 & 2725 \\ \hline
 & .... \\ \hline
\end{array}$$

From a careful study of the above example, we derive the following

**Rule for the extraction of the square root of a numerical quantity.**

(a.) *Set a point over the unit's figure, and over every alternate figure to the right and left.*

(b.) *Find, by inspection, the greatest number whose square is not greater than the number expressed by the figures in the first period.*

(c.) *Set this number in the quotient, and its square under the first period.*

(d.) *Subtract, and take down the two figures in the next period for a dividend.*

(e.) *Double the quotient for a new trial divisor.*

(f.) *Cut off the right hand figure of the dividend, and see how many times the trial divisor is contained in the number expressed by the remaining figures of the dividend.*

(g.) *Write the number in the quotient and in the divisor after the former figures.*

(h.) *Multiply the divisor thus formed by the last number in the quotient.*

(i.) *Subtract, take down the two figures of the next period, and repeat the processes* (e.), (f.), (g.), (h.), *until all the figures are taken down.*

## Examples worked out.

1. Find the square root of 430.5625

$$
\begin{array}{rl}
 & \dot{4}3\dot{0}.5\dot{6}2\dot{5}\;(\,20.75 \\
 & \underline{4} \\
40\,) & \overline{.30} \\
 & \underline{00} \\
407\,) & \overline{3056} \\
 & \underline{2849} \\
4145\,) & \overline{.20725} \\
 & \underline{20725} \\
 & \cdots\cdots
\end{array}
$$

*Note.* In extracting the square root of a decimal, or of a quantity made up of a whole number and a decimal, the number of decimal places in the root is the same as the number of periods or points in the decimal part of the original quantity.

2. Find the square root of $\frac{3}{5}$ to four places of decimals.

$$
\begin{array}{rl}
 & \frac{3}{5} = .6 \\
 & \dot{0}.6\dot{0}0\dot{0}0\dot{0}0\dot{0} \;(\,0.7745 \\
 & 49 \\ \cline{2-2}
147\,) & 1100 \\
 & 1029 \\ \cline{2-2}
1544\,) & .7100 \\
 & 6176 \\ \cline{2-2}
15485\,) & .92400 \\
 & 77425 \\ \cline{2-2}
 & 14975 \\ \cline{2-2}
\end{array}
$$

Ans. = .7745 correct to four places.

---

## Examples LXXX.

Find the square root of the following quantities:

1. 1681; 176464656; 19607184; 15129; 49000070000025; 217778088889; 2.178576; .00060516; 3.99960001.; $\frac{9}{16}$; $\frac{289}{1849}$; $1\frac{56}{169}$.

2. Extract to 5 decimal places the square roots of the following numbers:

2: .002; 13.5; 2.025; .0000007; 28.9; $\frac{1}{5}$; $\frac{2}{9}$; $3\frac{1}{4}$; $.\dot{5}$; $.0\dot{2}\dot{3}$; $5.\dot{6}$.

## APPLICATION OF THE METHOD OF FINDING THE CUBE ROOT TO NUMBERS.

Take as an example the number 12167.

Now $12167 = 12.10^3 + 1.10^2 + 6.10 + 7$

Treating this as an algebraical quantity, the process of finding the cube root will be as follows:

$$
\begin{array}{r|l}
 & 12.10^3 + 1.10^2 + 6.10 + 7 \;(2.10 + 3 \\
 & 8.10^3 \\ \cline{2-2}
(2.10)^2 \times 3 = 12.10^2 & 4.10^3 + 1.10^2 + 6.10 + 7. \\
2.10 \times 3 \times 3 = 18.10 & = 41.10^2 + 6.10 + 7 \\
3^2 = 9 & \\ \hline
13.10^2 + 8.10 + 9 & 39.10^2 + 24.10 + 27 \\ \cline{2-2}
 & 2.10^2 - 18.10 - 20 = 0 \\ \cline{2-2}
\end{array}
$$

The operation just given may be abridged as follows :—

$$\begin{array}{rcr|l}
 & & & 1\dot{2}16\dot{7}\ (\ 23 \\
 & & & 8 \\ \cline{4-4}
2^2 \times 300 & = & 1200 & 4167 \\
2 \times 3 \times 30 & = & 180 & \\
3^2 & = & 9 & \\ \cline{3-3}
 & & 1389 & 4167 \\ \cline{4-4}
 & & & \ldots \\ \cline{4-4}
\end{array}$$

From the above example the following Rule may be deduced for extracting the cube root of a numerical quantity :—

(a.) *Set a point over the unit's figure, and over every third figure to the right and left.*

(b.) *Find, by inspection, the greatest number whose cube is not greater than the number expressed by the figures of the first period.*

(c.) *Set this number in the quotient, and its cube under the first period.*

(d.) *Subtract, and take down the three figures in the next period for a dividend.*

(e.) *Square the quotient, and multiply it by* 300 *for a new trial divisor.*

(f.) *See how often this trial divisor is contained in the dividend.*

(g.) *Write this number in the quotient, and complete the divisor by* h, k, l.

(h.) *Multiply the product of the first and second parts of the quotient by* 30.

(k.) *Square the last figure of the quotient.*

(l.) *Add the three lines* e, h, k, *together, and multiply the sum by the last figure in the quotient.*

(m.) *Subtract the result from the dividend, and take down the next period, if there be one left.*

(n.) *Repeat the processes* e, f, g, h, k, l, m, *until all the figures are exhausted.*

## Examples worked out.

Extract the cube root of 885289075208.

| | |
|---|---|
| | $88\dot{5}28\dot{9}07\dot{5}20\dot{8}$ ( 9602 |
| | 729 |
| $9^2 \times 300 = 24300$ | 156289 |
| $9 \times 6 \times 30 = 1620$ | |
| $6^2 = 36$ | |
| 25956 | 155736 |
| $96^2 \times 300 = 2764800$ | ...553075 |
| | 000000 |
| $960^2 \times 300 = 276480000$ | 553075208 |
| $960 \times 2 \times 30 = 57600$ | |
| $2^2 = 4$ | |
| 276537604 | 553075208 |
| | ......... |

Extract the cube root of .03 to four places of decimals.

| | |
|---|---|
| | $.03\dot{0}00\dot{0}00\dot{0}00\dot{0}$ ( 3107 |
| | 27 |
| $3^2 \times 300 = 2700$ | 3000 |
| $3 \times 1 \times 30 = 90$ | |
| $1^2 = 1$ | |
| 2791 | 2791 |
| | 209000 |
| $31^2 \times 300 = 288300$ | 000000 |
| | 209000000 |
| $310^2 \times 300 = 28830000$ | |
| $310 \times 7 \times 3 = 6510$ | |
| $7^2 = 49$ | |
| 28836559 | 201855913 |
| | 7144087 |

*s.* = .3107, correct to four places of decimals.

*te.* There will always be as many decimal places in the root as are points over the decimal part of the quantity whose cube root uired.

### Examples LXXXI.

Find the cube roots of the following quantities :—

(1.) 778688 ; 59319 ; 2197 ; 21024576 ; 2017571115456 ; 86116663296.

(2.) 1601.613 ; 168.196608 ; .010764582912 ; .000000043243551 ; 1.345572864.

(3.) 1167575.877 ; .002325149908992 ; .252196389432.

4. Extract the cube root of the following to three places of decimals :—

12 ; .6 ; .04 ; $\frac{1}{4}$ ; .56 ; 1.25 ; $\frac{2}{3}$ ; $\frac{9}{64}$ ; $.\dot{7}$ ; $.\dot{4}\dot{5}$ ; $.9.\dot{6}$ ; 3.99960001.

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LONDON : PRINTED BY WILLIAM CLOWES AND SONS, STAMFORD STREET,
AND CHARING CROSS.

# ANSWERS

TO THE

# EXAMPLES IN FOWLER'S ALGEBRA.

180. c. 66.

BIBLIOTHECA BODLEIANA

# ANSWERS TO THE EXAMPLES.

## I.—Pp. 5, 6, 7.

17; 18; 54; 70; 0; 2; 20; 20; $-31$; $-8$; 13; 34; 12600 and 6804; 9; 25; 25; 5; $71\frac{1}{191}$; $1\frac{77}{150}$; $\frac{31}{67}$; 7; 7; 7; 14; 4; 10; 41; 0; 0; 9; 0; 51; 3; 1; 57; 2; 243; 27.

## II.—Pp. 9, 10, 11.

$2x+y$; $10a-16$; $11a^2x^3-7ax^2-5x$; $6x^3-48ax^2+32a^2x+6a^3$; $-1+2m-2n$; 0; $4(x^6+ax^5+a^2x^4+a^3x^3+a^4x^2+a^5x+a^6)$; 0; $x^5y-2xy^5$; $12+x$; $a^2+ab+b^2$; $3a^6-3a^4c^2$; $-7a^2b^2c^2-2a^2b^4+2a^2c^4+3b^4c^2+11b^2c^4+b^6-c^6$; $-10a-10b-10c-10d$; 0; 0; $3a^3+3b^3+3c^3-3abc$; $11m^3-2m^2n-22mn^2+18n^3$; $-29p^2qr-pq^2r+19pqr^2$; $\frac{47}{30}x^2+\frac{7}{4}xy+9y^2$; $\frac{1}{15}(61a^2-138b^2)$; $5am-7bn+6cp+9$; $13ayz+6bxz-cxy$; $\frac{55}{6}m-\frac{78}{15}n+\frac{25}{8}p+\frac{17}{4}q$; $a^3+b^3+c^3-3abc$.

## III.—Pp. 12, 13.

$x+4y$; $-5x$; $2x+2y$; $-6x$; $2x-2$; $x^2-3x+4$; $x^3-x^2$; $-x-4$; $3-x^2$; 6; $2b+2c$; $2a^3+a^2x+5ax^2$; $3a^3-8a^4+12a^5$; $2a^3+a^2x+2ax^2-x^3$; $2x^5y-7x^3y^3-2xy^5$; $a^2+a$; $8xy-6yz+4xz$; $-a^3+a^2+a+2$; $5x^3y-5x^2y^2-3xy^3$; $-5a+3b$; $4xy+2y$; $-y^2+1$; $-3x^2+4$; $-8x^3y-6x^2y^2-4xy^3+y^4$; $\frac{4b}{3}+\frac{7}{2}$; $\frac{1}{2}x-\frac{1}{2}y-a$; $-\frac{5x}{2}+\frac{3}{2}$; $\frac{1}{2}x^3+\frac{1}{2}x$; $\frac{1}{2}(a+b+c)$; $\frac{abc}{4}+2ab-\frac{1}{2}$; $\frac{1}{3}a^3-\frac{1}{3}a^2-a+1$; $p^2+\frac{2}{3}pq-\frac{1}{2}q^2$; $\frac{18}{7}a^2-2ab+\frac{7}{2}b^2$.

## IV.—P. 14.

$3-12x$; $5x^2-5y^2$; $7x^2+7y^2+7z^2$; $ab+ac$; $xy+xz-ax$; $2a+2ax+2ax^2$; $3x^3+3x^2+3x$; $4ab+4ac+4bc$; $a^2b+ab^2-ab$; $12a^2bc+12ab^2c+12abc^2$; $x^3yz-xy^3z-xyz^3$; $a^2+ab+ac$; $x^4y^2-x^2y^4$; $100x^2-50xy$; $11xyz-11abc$; $9ab^3+12b^4$; $3a^3b^3-3a^3b^2c-9a^2b^3c$.

## V.—P. 15.

$7a - 28x$; $2b$; $4ab$; $ax^2 - bx^2$; $3a^2b - 6ab + 3b$; $4x^3y - 4xy^3$; $a^3 - 2a^2b - ab^2$; $ax^2 - cx^2 - by^2 + dy^2$; $4bxy - x$; $-a^4 + 7a^2b^2 + ab^3 + 6b^4$; $12y^2 - x^2y^2$; $3a^2b^2 - 3b^2xy$; $0$; $2axy$; $3abx^2 - 3a^2xy + 3aby^2 - 3b^2xy$; $0$; $-x^2yz - xy^2z - xyz^2$.

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## VI.—P. 16.

25; 100; 36; 112; 268; 2; 60; 98; 2310; 74; 15; 2; 142; 0; 3; 237.

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## VII.—Pp. 19, 20, 21, 22.

(2.) $-15x^2y$; $x^3y^2z^3$; $49x^3y^3$; $-16x^3y^3$; $16x^3y^3z^3$; $-25a^2b^2x^2y^2$; $a^{14}b^{16}c^{15}$; $-6x^3y^4z^4$.

(3.) $2xy^3$; $-\frac{25}{7}b^2$; $-8a^2b^2c^2d^2$; $-\frac{2}{3}x^2y^2$; $\frac{25}{2}b^2c^2$; $-pqr^2$; $-3ab^4$; $\frac{9}{4}a^2b^2c$; $1$; $-2a^5c^5$; $3a^3$; $x^2y^2$; $\frac{abcd}{16}$; $a^3b^3$.

(5.) $2ab^2 + ab^3$; $-3a^2bc + 3ab^2c - 3abc^2$; $6a + 6b + 6c$; $-x^2y^2z - x^2yz^2 - xy^2z^2$; $3a^3b + 3a^2b^2 + 3ab^3$; $4x^6 - 12x^5y + 12x^4y^2 - 4x^3y^3$; $2x^3y - 2xy^3$; $-6x^2 + 4xy - 3y^2$; $-2a^4b^2c^2 + \frac{7}{5}a^6b^3c^2 + \frac{1}{5}a^2b^4c^2$; $8x^3 - 12x^2y + 6xy^2 - y^3$; $-\frac{1}{32}p^2q^2 + \frac{3}{16}pq^3 - \frac{1}{2}q^2$; $\frac{2}{9}x^4y^4z^5 - \frac{2}{9}x^3y^5z^5 - \frac{2}{9}x^3y^4z^6$; $2x^2 - 2x^4 - 2x^6$; $\frac{1}{4}p^4q^3r^3 + \frac{1}{4}p^4qr^5 + \frac{1}{4}p^2q^3r^5$.

(8.) $1 - x + x^2 - x^3 - 4x^4$; $x^4y^4 - 1$; $a - a^2 + 2a^3 + a^4 + 3a^5$; $a^4 - 8a^3 + 24a^2 - 32a + 16$; $4a^2 - b^2$; $a^5b^5 - 32$; $x^4 - 2b^2x^2 + b^4$; $a^4b^4 + a^2b^2 + 1$; $x^4 - y^4$; $a^5 - 5a^4p + 10a^3p^2 - 10a^2p^3 + 5ap^4 - p^5$; $a^5 - 2ax^4 - x^5$.

(9.) $9x^2 - 6x + 1$; $a^2 - 2ab + b^2$; $49x^2 + 70xy + 25y^2$; $a^4 + 2a^2 + 1$; $a^4 - 2a^2b^2 + b^4$; $1 - 2x^3 + x^6$; $a^2b^2 + 2abcd + c^2d^2$; $16a^2x^2 - 64abxy + 64b^2y^2$; $1 - 2x + 3x^2 - 2x^3 + x^4$; $x^4 - 2x^3 - x^2 + 2x + 1$; $b^2y^6 - 2aby^4 - 2by^3 + a^2y^2 + 2ay + 1$; $c^6 + 2c^4 - 2c^3 + c^2 - 2c + 1$.

(10.) $x^3 + 3x^2 + 3x + 1$; $a^3 + 3a^2b + 3ab^2 + b^3$; $1 - 3x + 3x^2 - x^3$; $8x^3 + 12ax^2 + 6a^2x + a^3$; $27x^6 + 27x^4y^2 + 9x^2y^4 + y^6$; $343a^3 - 147a^2 + 21a - 1$; $8a^3 + 36a^2b + 54ab^2 + 27b^3$; $x^6y^3 - 3x^5y^4 + 3x^4y^5 - x^3y^6$; $729 - 243a^2 + 27a^4 - a^6$; $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$.

(11.) $-84a^4b^2$; $a^4-b^4$; $x^4-4x^3+3x^2+4x-4$; $72(a^2xy-b^2xy)$;
$x^5-5x^3+4x$; $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$;
$x^6+2x^5y+2x^4y^2+2x^3y^3+2x^2y^4+2xy^5+y^6$;
$a^6-2a^5+2a^4-2a^3+2a^2-2a+1$; $a^5b^4+a^4b^5-a-b$;
$a^8-2a^6b^2+2a^2b^6-b^8$; 0; 0.

(14.) $4a^2-2ab+\frac{b^2}{4}$; $2a^3b+2ab^3$; $27a^2-9ab+3b^2$; $\frac{2}{3}x^2-\frac{13}{9}x+\frac{2}{3}$;
$a^4+\frac{5}{6}a^3+\frac{13}{36}a^2+\frac{5}{36}a+\frac{1}{36}$; $x^2-\frac{25}{12}xy+y^2$.

(15.) $\frac{1}{4}a^2-a+1$; $\frac{b^2}{9}+4b+36$; $\frac{a^2}{16}-\frac{ab}{6}+\frac{b^2}{9}$;
$p^2q^2-pq+\frac{1}{4}$; $a^4+a^3+\frac{3}{4}a^2+\frac{a}{4}+\frac{1}{16}$; $\frac{4}{9}a^2-2ab+\frac{9}{4}b^2$;
$9a^4-2a^2b^2+\frac{b^4}{9}$; $\frac{169}{144}x^2$; $\frac{1}{4}(y^2+2xy+x^2)$; $\frac{a^2}{4}+\frac{ab}{3}+\frac{ac}{4}$
$+\frac{b^2}{9}+\frac{bc}{6}+\frac{c^2}{16}$; $9a^4-6a^3b-a^2b^2-\frac{2ab^3}{3}+\frac{b^4}{9}$;
$\frac{25}{36}x^4-\frac{5}{3}x^3y+3x^2y^2-\frac{12}{5}xy^3+\frac{36}{25}y^4$.

(16.) $a^3-\frac{3a^2}{2}+\frac{3a}{4}-\frac{1}{8}$; $27a^6-\frac{27}{4}a^4+\frac{3}{16}a^2-\frac{1}{64}$;
$\frac{1}{8}(a^3+3a^2b+3ab^2+b^3)$; $\frac{x^3}{27}-\frac{x^2y}{6}+\frac{xy^2}{4}-\frac{y^3}{8}$;
$x^3-\frac{3x^2y}{2}+\frac{3xy^2}{4}-\frac{y^3}{8}$; $8(x^3-3x^2y+3xy^2-y^3)$.

## VIII.—Pp. 24 to 28.

(1.) $6a$; $6a$; $-5b$; $-5a$; $8a^2b^2$; $a^2c$; $5x$; $100x^3y$; $-7a^3$; $-10abc$; $a$.

(3.) $50y$; $\frac{8}{3}$; $-\frac{289}{2}a$; $\frac{3}{8}y$; $2x^2$; $\frac{4}{25}a^2$; $2a$; $9b$; $-\frac{x^2}{5}$; $-\frac{ax^2y^2z}{2}$;
$-\frac{10}{5}ab$; $\frac{3}{65}cd$; $\frac{2x}{3}$; $2400ap^3$; $\frac{81}{250}bpr$.

(4.) $3(a^2+b^2)$; $a-2b$; $x-y+z$; $a-2b+c$; $\frac{3}{2}a-\frac{b}{2}$;
$\frac{x^3}{4}-\frac{x^2y}{2}+\frac{xy^2}{4}-\frac{y^3}{2}$; $\frac{m^2}{2}+\frac{mn}{4}+\frac{n^2}{6}$; $10m^2-m+\frac{1}{10}$;
$\frac{3a^4}{4b^2}-\frac{3}{2}\frac{a^3}{b}+b^2$; $\frac{p^2}{4}+pq$; $\frac{a^2}{2}+\frac{3ax}{2}+\frac{x^2}{2}$; $4x^2-2y^2+4z^2$;
$27x^2y-3x^2y+\frac{1}{3}yz^2$.

(8.) $x - 1$; $x + 1$.

(9.) $2a^2 + a$; $x + 4$.

(10.) $x^3 + x^2 + x + 1$; $x^2 - 2x + 4$.

(11.) $x^2 - x - 3$.

(12.) $7x^2 - 8x + 4$.

(13). $a - 2b$.

(14.) $a + 2$.

(15.) $a^3x^3 + a^2x^2y + axy^2 + y^3$; $10a - b$.

(16.) $x^2 + xy + y^2$.

(17.) $xyz + 5$.

(18.) $2ab - 3cd$.

(19.) $2a + 6$, &c.

(20.) $a + b - c$; $a + b + c$; $-a - 3b - c$, &c.; $a + 3b - c$, &c.

(21.) $a - b - c$.

(22.) $2a^2 - 4a + 3$.

(23.) $x - 2y + 3z$.

(24.) $a - pq$.

(28.) $x - x^2 + x^3 - x^4$, &c.

(29.) $3 + 3x + 3x^2 + 3x^4$; $6 + 7a + 7a^2 + 7a^3$, &c.

(30.) $ax + ax^2 + ax^3 + ax^4$, &c.

(31.) $1 + x + x^2 + x^4$, &c.

(32.) $1 + 2x^3 + 2x^6 + 2x^9$, &c.; $x^6 + x^5 + x^4 + x^3$, &c.; $a^5 + 3a^4 + 9a^3 + 27a^2$, &c.; $1 - a + a^2 - a^3$, &c.; $x^3 + 5ax^2 + 11a^2x + 15a^3$, &c.; $(x - a)^3$; $1 + 5x + 12x^2 + 19x^3$, &c.; $x^4 - x^3 + 2x - 2$, &c.

(35.) $x + \frac{1}{2}$; $x - \frac{1}{3}$; $a - \frac{1}{3}$; $\frac{3}{2}a + \frac{3}{2}x$; $\frac{3}{2}m^2 - \frac{m}{2} + \frac{1}{6}$; $\frac{a^2}{2} - 1$; $\frac{a}{2} + b$.

(36.) $6a - 6$; $\frac{x - y}{2}$; $a^4 + 3a^3 + 9a^2$, &c.; $a - \frac{2}{3}$; $\frac{2}{3}p^2q - \frac{3}{2}pq^2$; $\frac{a^2}{4} - \frac{ab}{2} + b^2$.

(37.) $3 - 9x + 27x^2 - 81x^3$, &c.; $1 + \frac{1}{6x} - \frac{1}{18x^2} + \frac{1}{54x^3}$, &c.; $1 - \frac{x}{3} + \frac{10x^2}{9} - \frac{10x^3}{27}$, &c.; $1 + \frac{x}{2} + \frac{x^3}{2} + \frac{x^3}{4}$.

(40.) $x + a - 1$. (41.) $a^2 + b^2 + c^2 - ab - ac - bc$. (42.) $a + b + c$.

(43.) $x^2 - (a + 1)x - 1$. (44.) $a^2 - 2ab + b^2 - c^2$.

(45.) $x^2 + (a + b)x$, &c. (46.) $a - p$.

(47.) $(a + b + c)(a + c - b)(a + b - c)$;
$(a + b + c)(a + c - b)(b + c - a)$;
$(a + b + c)(a + b - c)(b + c - a)$;
$(a + b - c)(a + c - b)(b + c - a)$.

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## XI.—P. 34.

(2.) $4a^2$; $3x$; $5mnp$; $6a^2b^4c^5$; $7ax^2$; $8a^3b^6$; $a^3b^4cd^5$.

(3.) $a$; $2a$; $2abc$; $2a^3b^{\frac{4}{3}}c^{\frac{2}{3}}$; $5a$; $6xy^2z^3$; $7ab^4c^8$; $8mn^2p$; $9z^5$; $3mn$; $4xy^3$.

(4.) $2ab$; $3ab^2c^3$; $xyz^2$.

(5.) $3a$; $a^2x^2$; $4xy^2$; $2ab^2c^3$; $1$; $1$; $2a$; $2a^3$; $3mn^4$; $7a^2x$; $11xy^3z^5$.

(7.) $\frac{a}{3}$; $\frac{3a}{4b}$; $\frac{1}{5ab}$; $\frac{3x}{10}$; $\frac{4xy^2z^3}{9ab^2c^3}$.

(8.) $\frac{2}{3}a\sqrt[3]{225}$; $\frac{3}{2}a^2b^3$; $\frac{4x^2y^4}{3a^4b^2}$; $\frac{x^2}{3}$; $\frac{2}{3abc}$; $\frac{13a}{14c}$.

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## XII.—Pp. 36, 37.

(1.) $a - 2b$; $2a + b$; $x - 3$; $2x - 3$; $x^2 + 1$; $ax - 2$; $a^3 - 5$;
$2a^2b + 7$; $3a^4 - 4b$; $5x^2 + 6y$; $4ab + 5cd$; $6x^3y^2z - 7$;
$12xz^2 - 3xy^2$; $7ad - 11bc$.

(3.) $a + b - c$; $a - 2b - 3c$; $3x^3 - 3x - 1$; $3a^3 - 2a^2 + 7a + 11$;
$x^6 - x^4y^2 + x^2y^4 - y^6$.

(4.) $a - \frac{b}{2}$; $2a^2 - \frac{a}{3}$; $\frac{2}{5}x - 1$; $\frac{2}{3}x + \frac{3}{2}y$; $\frac{x^2}{2} - \frac{xy}{4} + \frac{y^2}{2}$; $2a^2b - \frac{ab^2}{2}$;
$\frac{a^2}{2} + ab + \frac{b^2}{3}$.

(6.) $2a^m - 3$; $a^p - 2b^q$; $a^{2p} \quad 3a^p + 3$; $3a^nb^m - 4a^mb^n$;
$x^my^2 - 4x^2y^n$; $\frac{2xy^2}{5} + \frac{5x^2y}{3} - \frac{3x^3}{2}$.

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## XIII.—P. 39.

(1.) $a + 1$; $x - 1$; $x - 2$; $a^2 - 3$; $4x + 1$; $2a + 3x$; $1 - 5ab$;
$-2x^2 - 4x$.

(3.) $x^2 + 3x - 7$; $x^3 + 3x^2y + 3y^2x + y^3$; $3a - 2b - c$; $a^3 - a^2 + a - 1$;
$a^3 + 3a^2b^2 + ab + 3b^3$; $\frac{a}{2} - 3$; $\frac{2}{3}x - \frac{3}{2}y$; $4a^2x - \frac{1}{4}ax^2$;
$a^{2n} - a^n + 1$; $x + y - 2z$.

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## XV.—Pp. 43, 44, 45.

(2.) $x + 3$; $x - 8$; $x^2 - 3$; $x + 1$; $x - 4$; $2x + 1$;
(4.) $a + b$; $x - 2$; $x - 1$; $(x - 1)^2$; $x - 4$; $x - 10$; $x^2 - 4x + 5$;
$7x - 3$; $x^2 + 4$; $x^2 + 5x + 3$; no G. C. M.; ;
; $x^2 - 7x + 10$; $a - 2b$;
(6.) $x^2 - 2$; $3x - 8$; $a - 3b$; $11a - 10b$; $4a - 3b$; $a + 2b$;
(7.) $2(x + 1)$; $xy(x^2 + y^2)$; $2(x - 2y)$; $x - 4$; $x - 1$; $y(7x - 3)$;
$x - 6$.

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## XVI.—Pp. 46, 47.

$2x$; $a^2bc$; $2$; $24xyz$; $abcd$; $16p^3q^4r^3$; $xy$; $x + y$; $x^2$; $3x$; $(a + b + c)$;
$x^2 + y^2$; $x + 3$; $2x + 1$; $x - 10$; $3a - 2b$; $x^2 + 4$; $x + 1$; $x + 3$.

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### XVII.—P. 48.

The answers to these may be at once verified by taking the G.C.M. of the quantities as given in the answers to XV. and XVI., and performing the operation of Art. (49).

### XIX.—P. 52.

In these answers *one* only of the factors is given.

$(a-1)$; $(x-1)$; $(2x-3y)$; $2x-3$; $x-y^2$; $2ax-3b^2$; $3ab^2c^3-1$; $4p-q$; $3ax-4b^2$.

$x-a$; $2a-x$; $ax-by$; $16(2x-y)$; $9a^2b^2c^2+xyz$; $2+a$; $a+b$; $2a-1$; $pq^2r^3-3$; $6x^2-3$; $x-y$; $x+y$; $4a^2(a-x)$; $2(2x+1)$; $a^4(a-x)$; $3(a+2)$; $x^3(a-x)$; $2$; $p^3(p+2q)$; $4x^2(x+2a)$.

### XX.—P. 53.

$(x+2)$;
$x+7$;
$x-10$;
$x+6$;
$x+2y$;
$p+5q$;
$x(x-7)$;
$x+y$;
$4ab(a-9b)$;
$x^2(by-9ax)$;
$x^2y^2(x+7y)$;

$x+12$;
$x-7$;
$x-7$;
$x+6$;
$x-4y$;
$p-4q$;
$x^2(x+11)$;

$x+10$;
$x-11$;
$x+7$;
$x-6$;
$x+5y$;
$p-6q$;
$y(y-3)$;
$b^2(a+20b)$;
$(x^2+9y)$;
$a^3-9b^2$;
$xyz+10$.

### XXI.—P. 55.

$$\frac{a^2+b^2}{a};\ \frac{a-b}{a};\ \frac{a^2-b^2}{b^2};\ \frac{a^2c^2-4b^2}{c^2};\ \frac{2x}{x+y};\ \frac{2y^2}{x^2-y^2};$$

$$\frac{y^2-xy}{x+y} - \frac{a^2-4ab+b^2}{2(a-b)}.$$

$$\frac{14a^2-6ab+8b^2}{7a+8b};\ \frac{2a(x^3-a^2x+a^3)}{x+a};\ \frac{4x^3-x^2+x+1}{x^2-1};\ -\frac{x}{x+1};$$

$$\frac{2(a^2-ab+bc)}{a+b+c}.$$

### XXII.—P. 56.

(1.) $abc$; $\dfrac{x^2y^2}{abc}$; $\dfrac{a}{c}$; $\dfrac{a^2+x^2}{a}$; $\dfrac{4(a^4+a^3x-ax^3-x^4)}{a^2+x^2}$.

(2.) $\dfrac{xy}{b^2c}$; $\dfrac{1}{x-y}$; $\dfrac{x^2+xy+y^2}{x^2+y^2}$; $\dfrac{a+x}{a-x}$; $\dfrac{7abcxy}{a^2b^2-c^2y^2}$.

(3.) $\frac{10\,a(a^2-b^2)}{b}$; $\frac{9abc}{7}$; $\frac{3x^2y^2}{2}$; $\frac{(a-b)^3}{ab}$; $\frac{3\,(x+y+z)^2}{(x-y+z)^2}$; $\frac{(x+y)^2}{(x-y)\,(x^2-y^2)}$;
$\frac{abxy}{x^6-y^6}$; $\frac{x^2\,(x^3+a^3)}{x^3-a^3}$.

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## XXIII.—P. 58.

$\frac{3x}{4yz}$; $\frac{ab}{2x^2y}$; $\frac{6a^2b^2}{7c^2d^2}$; $\frac{5abcd}{14}$; $\frac{1}{a+1}$; $\frac{2}{3\,(a-b)}$; $\frac{x^2-1}{3}$; $\frac{y+a}{5b-2c}$;
$\frac{x^2-x+1}{x^2(x+1)}$; $\frac{a+1}{2\,(a^2+a+1)}$; $\frac{9abc}{4\,(a+b+c)}$; $\frac{x-7}{x+3}$; $\frac{x-y}{x^2+y^2}$;
$\frac{x-2}{2x-1}$; $\frac{x}{x+b}$; $\frac{a+b}{a-b}$; $\frac{x^4+a^4}{x^4+ax^3+a^2x^2+a^3x+a^4}$; $\frac{pqr}{p+q+r}$;
$m^2-n^2$ will not reduce.

The two last examples in xxiii. should be placed in xxiv.

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## XXIV.—P. 59.

$1+\frac{2b}{a-b}$; $5x+5+\frac{5}{x-1}$; $a+b+\frac{2b^2}{a-b}$; $x+9+\frac{66}{x-7}$;
$x+3y-\frac{y^2}{x+y}$; $x+\frac{y^2+yz+z^2}{x+y+z}$; $a^2-ab+b^2-\frac{2b^3}{a+b}$;
$4x+3y+\frac{7y^3}{3x^2+7y^2}$; $a^2+2ab+3b^2+\frac{4b^3}{a-b}$; $a-3b+\frac{2b^2\,(a+b)}{a^2+b^2}$.

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## XXV.—P. 60.

$\left(\frac{ab}{b^2}, \frac{a^3}{b^2}\right)$; $\left(\frac{6x^2}{12x^3}, \frac{4x}{12x^3}, \frac{3}{12x^3}\right)$; $\left(\frac{ab^2c}{a^2bc}, \frac{ab^3}{a^2bc}, \frac{a^4}{a^2bc}, \frac{b^2c^2}{a^2bc}\right)$; $\left(\frac{a^2}{abc}, \frac{b^2}{abc}, \frac{c^2}{abc}\right)$;
$\left(\frac{d}{abcd}, \frac{c}{abcd}, \frac{a}{abcd}, \frac{b}{abcd}\right)$; $\left(\frac{bcy^2z^2}{x^2y^2z^2}, \frac{acx^2z^2}{x^2y^2z^2}, \frac{abx^2y^2}{x^2y^2z^2}\right)$;
$\left(\frac{b\,(a^2-b^2)}{ab\,(a^2-b^2)}, \frac{a\,(a^2-b^2)}{ab\,(a^2-b^2)}, \frac{ab\,(a-b)}{ab\,(a^2-b^2)}, \frac{ab\,(a+b)}{ab\,(a^2-b^2)}\right)$;
$\left(\frac{6ax+6x^2}{12x}, \frac{4ax-4x^2}{12x}, \frac{3a^2-3x^2}{12x}\right)$; $\left(\frac{30a}{60}, \frac{20b}{60}, \frac{15c}{60}, \frac{12d}{60}\right)$;
$\left(\frac{xz-yz}{xyz}, \frac{xy-xz}{xyz}, \frac{xy-yz}{xyz}\right)$;
$\left(\frac{(x^2+x+1)\,(x+1)}{(x^3-1)(x-1)}, \frac{x\,(x^2+x+1)}{(x^3-1)\,(x+1)}, \frac{x^2\,(x+1)}{(x^3-1)(x+1)}\right)$;
$\left(\frac{a^2\,(x^2+y^2)}{x^4-y^4}, \frac{a^2\,(x^2-y^2)}{x^4-y^4}, \frac{a^4}{x^4-y^4}\right)$;
$\left(\frac{xy(y-z)(x-z)}{2(x-y)(y-z)(x-z)}, \frac{yz(x-y)(x-z)}{2(x-y)(y-z)(x-z)}, \frac{xz(x-y)(y-z)}{2(x-y)(y-z)(x-z)}\right)$;
$\left(\frac{pm^2}{m^2(m-n)}, \frac{mpq}{m^2(m-n)}, \frac{pq^2}{m^2(m-n)}\right)$; $\left(\frac{x^3+y^3}{x^4+x^2y^2+y^4}, \frac{x^3-y^3}{x^4+x^2y^2+y^4}\right)$;

$$\left(\frac{(x+a)^3(x^2+a^2)^2}{(x^2-a^2)^3(x^2+a^2)^2}, \frac{(x-a)^3(x^2+a^2)^2}{(x^2-a^2)^3(x^2+a^2)^2}, \frac{(x^2-a^2)}{(x^2-a^2)^3(x^2+a^2)^2}, \frac{(x^2-a^2)^3}{(x^2-a^2)^3(x^2+a^2)^2}\right); \left(\frac{(x^4+a^4)^2}{x^8-a^8}, \frac{(x^4-a^4)^2}{x^8-a^8}\right);$$

$$\frac{x(x^2+1)(x^2-x+1)(x^4+1)}{(x^2+1)(x^3+1)(x^4+1)}, \text{\&c.}; -\frac{6a(a-x)^2}{12(a-x)^3}, \text{\&c.}$$

## XXVI.—P. 61.

(1.) $\frac{13a}{12}$; $\frac{2a+b+c}{10}$; $\frac{3a^2+2a^2b+ab^2+4b^2}{a^2b^2}$;

$$\frac{2a^3-a^2x+4ax^2-3x^3}{ax(a-x)}; \frac{x^5+3ax^4+a^2x^3-a^3x^2}{a(x^4-a^4)}; \frac{2a^3}{a^4+a^2b^2+b^4};$$

$$\frac{3a^2+b^2}{a^3-b^3}; \frac{x^3y+2x^2y^2+3xy^3+4y^4}{x^4}; \frac{3x^3+4}{x^4-1};$$

$$\frac{ax^5-3a^2x^4+5a^3x^3-3a^4x^2+a^5x}{(x-a)^6}; \frac{3x^2+32x+63}{(x+7)(x+9)};$$

$$\frac{x^2+3x+1}{x^2+5x+6}; \frac{2(ab+ac+ad+bc+bd+cd)}{abcd}.$$

(2.) $\frac{x}{6}$; $\frac{16a^2-b^2}{4ab}$; $\frac{x^2-ax}{a^2}$; $\frac{2x}{a^2-x^2}$; $\frac{ax}{a^2-x^2}$; $-\frac{2}{x+3}$; $\frac{1}{x^2-7x+12}$;

$$-\frac{8x-42}{x^2-x-42}; \frac{x^3-15x}{x^4+7x^2+10}; \frac{(a^2+a)(x-1)-2}{a^2x^2-1};$$

$$\frac{9x^2+9x-1}{12(x^3-1)}.$$

(3.) $\frac{3x^2+4x-5}{x^5}$; 0; $-\frac{2x(x+a)}{(x-a)^2}$; $\frac{6a+6b+4c+4d}{(a+b)^2-(c+d)^2}$; $\frac{x-a-4}{x^2-a^2}$;

$$\frac{x^4+4ax^3-2a^2x^2+4a^3x+a^4}{x^4-a^4}; \frac{2b}{a-c};$$

$$-\frac{20a^3+10a^2x-3a^2+10ax^2+5x^3}{15(a^3-x^3)}.$$

## XXVII.—P. 63.

(3.) $\frac{a^3b^2}{c^3}$; $\frac{4bc^3}{15ad^2}$; $\frac{x^3-a^3}{x^3+a^3}$; $\frac{a^2b}{c(x^2-a^2)}$; $\frac{7}{8y^2}$; $\frac{4}{7ax^2(3x-2)}$;

$$\frac{1}{6x^2+31x+35}; \frac{1}{27x^5-72x^3-33x^2+88}; \frac{1}{x^4+a^2x^2+a^4}.$$

(5.) $\frac{4b(2a-b)}{a^2-b^2}$; $\frac{a^3+2a^2b+2ab^2+b^3}{b(a^3+2a^2b+2ab^2-b^3)}$; $\frac{(a-b)^2}{3b^2(a+b)}$; $\frac{x(a+x)}{x+y}$;

$$\frac{6x^2}{a(a^2+x^2)}; \frac{1+a^2}{2a(1-a^2)}; \frac{p+2q}{3p}; 1; \frac{2}{(2a-b)^2}; \frac{1}{1+x^2};$$

$\frac{1+2x+2x^3-x^4}{(1-x)^4}$; $\frac{(a-b)(a^2+b^2)}{(a-3b)(a^2+b)}$.

## XXVIII.—P. 66.

(1.) $\frac{1}{2}$; $\frac{a^3}{b^3}$; $\frac{5x}{4y}$; $\frac{49y}{16b}$; $\frac{x}{2y}$; $\frac{9y}{16x}$; $-\frac{27b}{22a}$; $\frac{abc}{xyz}$; $-\frac{3x^4}{y^4}$; $-\frac{2bc^4d^7}{3p^2q^4r^6}$; $\frac{28a^2b^2}{45xy}$.

(2.) $\frac{a+b}{b-a}$; $\frac{2y(a+b+c)}{x(a+b-c)}$; $-\frac{9(ax+by)}{px+qy}$; $\frac{3(a+2b+3c)}{2b(a-2b+3c)}$; $\frac{x^2+1}{2}$; $\frac{16x}{1-x^2}$; $\frac{y(x^2-a^2)}{a^2b}$.

(4.) $\frac{4ay}{27bx}$; $\frac{9abc^2}{2b-5a}$; $\frac{x+1}{x-1}$; $\frac{x+4}{2x+4}$; $x^4+x^2+1$; $1$; $\frac{x^2+4y^2}{(x+2y)(x^2-4y^2)}$; $\frac{x^5+2x^4-x-2}{3(x^4+1)}$; $\frac{4(x+7)}{7x(x-3)}$; $\frac{x^7+x^5-2x^4+x^3-2x^2+2}{x^3+2}$.

(5). $\frac{6(1-a)}{a(1+a)}$; $a(a^2+1)$; $\frac{2x^2}{2x+1}$; $\frac{3a^2}{4}$; $\frac{1+x-6x^2}{4}$; $-\frac{x^2+y^2}{y^2}$; $\frac{a}{b}$; $\frac{1}{2ab}$; $\frac{3cdx}{2b^2y}$; $\frac{a+x}{6(a-b)}$; $\frac{1}{x^2-a^2}$; $\frac{a}{a-x}$; $\frac{x^2+x+1}{x^3+1}$; $abc^4(a^2-b^2)$; $\frac{7(a^2+3b^2)}{3a^2(a^2-7b^2)}$.

## MISCELLANEOUS.—P. 67.

(1.) $\frac{a+b}{ab}$; $\frac{b-a}{ab}$; $\frac{1}{ab}$; $\frac{b}{a}$; $\frac{a^2+ab+b^2}{ab(a+b)}$.

(2.) $\frac{bc+1}{abc+a+c}$; if $a > b$, then $\frac{a+b}{a^3-b^3}$ is greatest, and $\frac{1}{a^2+ab+b^2}$ is least.

(3.) $\frac{y}{x^2(y-x)}$.

(5.) $\frac{1}{(x-a)(x-b)(x-c)}$.

(6.) $\frac{a^2+b^2}{4a^3-2ab^2}$.

(7.) $\frac{x}{4x^2-y^2}$; $\frac{3x}{(x+1)(x+2)}$; $\frac{2}{x^3-1}$.

## XXIX.—P. 72.

1. $x = 3$.
2. 7.
3. 8.
4. 3.
5. 1.
6. 3.
7. $2\frac{1}{4}$.
8. 2.
9. 2.
10. $3\frac{1}{3}$.
11. 1.
12. 6.
3.
6.
2.
12.
$-\frac{1}{2}$.
4.
7.
$-2$.
$-5$.
$-2b$.
$-2.\ \frac{ab+ac+bc}{a+b+c}$
$\frac{ab-cd}{a+b-c-d}$
$\frac{2}{3}$.
2.
2.
13. $x = 2$.
14. 2.
15. 6.
16. 90.
17. 1.
18. $-2$.
19. 1.
20. 4.
21. $7\frac{1}{3}$.
22. 32.
23. 7.
24. 7.
25. 7.
26. $12\frac{16}{43}$.
27. 6.
28. $6\frac{12}{25}$.
29. $\frac{5}{3}$.
30. $\frac{3}{2}$.
31. 55.
32. $8\frac{141}{218}$.
35. $1\frac{3}{7}$.
36. 7, 2.
37. $1\frac{9}{11}$.
38. 7, 4.
39. 12, 27.
40. $7\frac{4}{23}$.
41. $= 57\frac{6}{11}$.
42. $x = 10\frac{5}{51}$.
72.
$\frac{a+b+c}{d}$
2.
$\frac{11}{13}$.
$\frac{2}{3} \cdot \frac{5c-6d}{10a-5c+}$
44. $1\frac{3}{11}$.
45. 2.
46. $-6\frac{2}{3}$.
47. 3.
48. $-7$.
49. $\frac{5a}{4}$.
50. $a$.
51. 19.
53. $\frac{d(a+c)}{b}$, 1
54. 8, $\frac{a(b-c)}{b-a}$
55. 51.
56. $\frac{(a^2+b^2)(a+b)+}{a+b-ab}$
57. 2.
58. $-1$.

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## XXX.—P. 77.

1. 7.
2. 25.
3. 3*s*. 9*d*. and 3*s*. 3*d*.
4. $9\frac{2}{3}$, $12\frac{2}{3}$, $14\frac{2}{3}$.
5. 3 P.M., $31\frac{1}{2}$ miles.
6. 8 and 24.
7. 100 lbs.
8. 46 lbs.
9. 18, 20, &c.
10. 5, 5, 5, 10.
11. 4, 6, 8.
12. 740.
13. 4 days after.
16. $4\frac{32}{37}$ minutes.
17. $10\frac{70}{73}$ hours, $104\frac{8}{73}$ miles.
18. £1928 11*s*. $5\frac{1}{4}$*d*.
19. 3 days after.
20. 400, 200, 600.
21. 3969.
22. $\frac{7a-6n}{2}$
23. 9 and 3.
24. 25 and 50.
25. 5 acres.
26. 14 and 6.
27. 100 apples, 24 oranges.
28. $20\frac{1}{2}$ and $85\frac{1}{2}$.
29. 49′ $\frac{4}{11}$ past 9.
32′ $\frac{8}{11}$ past 9.
5′ $\frac{5}{11}$ past 10.
38′ $\frac{2}{11}$ past 10.
6.
30. 190 and 191.
31. 34 oxen, 1 sheep
32. 200 guineas.
33. 560 and 240 guineas.
34. 14 from Exeter, t
second meeting.
35. £437 19*s*. $0\frac{84}{127}$*d*.

36. 10 from the 1st.
4 „ 2nd.
37. 36 miles.
38. 2 miles an hour.
39. 4*d.*
40. 10, 21, 43.
41. distance = 60 miles, rates 20 and 30.
42. distance = 150 miles. rate = 40 miles.

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## XXXI.—P. 85.

(1.) $x=1$
$y=3.$
(2.) $x=4.$
$y=7.$
(3.) $x=480.$
$y=672.$
(4.) $x=10.$
$y=1.$
(5.) $x=8.$
$y=3.$
(6.) $x=7\frac{16}{23}.$
$y=4\frac{1}{23}.$
(7.) $x=7.$
$y=2.$
(8.) $x=6.$
$y=4.$
(10.) $x=a+b.$
$y=\frac{a-b}{6}$
(11.) $x=9.$
$y=6.$
(12.) $x=3.$
$y=-4.$
(14.) $x=2.$
$y=3.$
(15.) $x=8.$
$y=4.$
(16.) $x=480.$
$y=672.$
(17.) $x=4\frac{21}{232}.$
$y=3\frac{23}{29}.$
(18.) $x=\frac{ad-bc}{dm-bn}$
$y=\frac{ad-bc}{dm-bn}$
(19.) $x=\frac{m^2-n^2}{am-bn}$
$y=\frac{n^2-m^2}{an-bm}$
(20.) $x=\frac{b^2-a^2+c}{2a}$
$y=\frac{a^2-b^2+c}{2b}$
(21.) $x=0.$
$y=1.$
(22.) $x=c.\frac{a^2+b^2}{a^2-b^2}$
$y=c.\frac{a^2+b^2}{2ab}$
(23.) $x=15.$
$y=99.$
(24.) $x=\frac{ab(a+b)}{3a^2+4ab+3b^2}$
$y=-\frac{ab(a+b)}{2(a^2+ab+b^2)}$
(25.) $x=\frac{a^2+ab-b^2}{2a}$
$y=\frac{a^3-b^3}{2a(a+b)}$
(26.) $x=\frac{a}{b+c}$
$y=-\frac{a}{b+c}$
(27.) $x=\frac{a^2}{a-b}$
$y=-\frac{b^2}{a-b}$
(28.)
(29.) $x=\frac{b'c-bc'}{ab'-a'b}$
$y=\frac{ac'-a'c}{ab'-a'b}$
(30.) $x=8\frac{68}{89}.$
$y=12\frac{24}{89}.$
(31.) $x=5.$
$y=5.$
(32.) $x=-8\frac{3}{7}.$
$y=-18\frac{1}{7}.$

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## XXXII.—Pp. 88 and 89.

(1.) $x=3$, $y=5$, $z=7$
(2.) $x=10$, $y=2$, $z=3$
(3.) $x=1$, $y=2$, $z=3$
(4.) $x=2$, $y=3$, $z=4$
(5.) $x=8$, $y=6$, $z=4$
(6.) $x=1$, $y=0$, $z=2$
(7.) $x=24$, $y=40$, $z=56$
(8.) $x=1$, $y=0$, $z=0$
(9.) $x=2$, $y=3$, $z=4$
(10.) $x=1$, $y=0$, $z=0$
(11.) $x=24$, $y=60$, $z=120$
(12.) $x=\frac{b^2+c^2-a^2}{2bc}$, &c.

(14.) $x = \frac{1}{(a-c)(b-c)}$ &c.; (15.) $\left.\begin{array}{l} x = 3 \\ y = 5 \\ z = 7 \end{array}\right\}$ (16.) $x = \frac{2ac}{c-a}$ &c.

(17.) $\left.\begin{array}{l} x = 4 \\ y = 5 \\ z = 6 \end{array}\right\}$ (18.) $\left.\begin{array}{l} x = -6\frac{6}{23} \\ y = 3\frac{21}{41} \\ z = 2\frac{2}{41} \end{array}\right\}$ (19.) $\left.\begin{array}{l} x = 6 \\ y = 1 \\ z = 3 \end{array}\right\}$

(20.) $\left.\begin{array}{l} z = 2(a+b+c) \\ y = 4(ab+ac+bc) \\ x = 8abc \end{array}\right\}$

## XXXIII.—Pp. 89, 90, 91.

(3.) $\left.\begin{array}{l} x = 7 \\ y = 2 \end{array}\right\}$ (4.) $\left.\begin{array}{l} x = 5 \\ y = 18 \end{array}\right\}$ (5.) 15 and 42. (6.) $40\frac{1}{2}$; 81; $94\frac{1}{2}$. (7.) 2*s*. 6*d*. and 1*s*. 9*d*. (8.) 6. (9.) Set wrong. (11.) 24. (12.) 10 and 30. (14.) 60. (15.) 96 and 144. (16.) 200 and 1200. (17.) (18.) 12; 18; 24 hours. (19.) $28\frac{8}{19}$; $66\frac{6}{19}$. (20.) 50 and 30 gallons. (21.) $2\frac{10}{27}$. (22.) £24 and £12. (23.) 12; 10; 11. (24.) 40 and 21. (25.) 10 and 6 hours. (26.) 3850 artillery, 1750 cavalry, 4550 infantry.

## XXXIV.—Pp. 94, 95.

$\left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$; $\left(\frac{2a}{b}-\frac{b}{2a}\right)\left(\frac{2a}{b}+\frac{b}{2a}\right)$; $(2a-3b)(2a+3b)(4a^2+9b^4)$;

$(4xyz-1)(4xyz+1)(16x^2y^2z^2+1)$; $\left(\frac{x}{3}-\frac{3}{x}\right)\left(\frac{x}{3}+\frac{3}{x}\right)$;

$\left(x-1+\frac{1}{x}\right)\left(x+1+\frac{1}{x}\right)$; $(a^2+3ab+b^2)(a^2+ab+b^2)$;

$(x+3)(x-1)$; $(x-1)^2(x+1)^2x^2(x^2-2)$;

$\left(\frac{1}{2x}-\frac{x}{3}\right)\left(\frac{1}{2x}+\frac{x}{3}\right)\left(\frac{1}{4x^2}+\frac{x^2}{9}\right)$; $(2a+b+c)(b+c)$;

$(x^2y^2-y^2z^2-x^2z^2)(x^2y^2-y^2z^2+x^2z^2)$;

$\left(\overline{\frac{x^2}{y}+\frac{y^2}{x}}\Big|^2-\overline{\frac{x^2}{z}+\frac{z^2}{x}}\right)\left(\overline{\frac{x^2}{y}+\frac{y^2}{x}}\Big|^2+\frac{x^2}{z}+\frac{z^2}{x}\right)$;

$\frac{(2y^2-x^2)(2y^2+2xy+x^2)}{y^2(y+x)^2}$; $\frac{(41a^2-18ax+41x^2)(82ax-9a^2-9x^2)}{400(a-x)^2(a+x)^2}$;

$\frac{(a+b+c)(a+b-c)(b+c-a)(a+c-b)}{4b^2}$; $\frac{(bx-ay)^2(bx+ay)^2}{a^4b^4}$.

## XXXV.—P. 95.

$(x-2y)(x+2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$;

$(2a^2+3b^2)(4a^4-6a^2b^2+9b^4)$; $(5x+1)(25x^2-5x+1)$:

$(2a - b)(2a + b)(4a^2 - 2ab + b^2)(4a^2 + 2ab + b^2)$;
$(4a^2b + 3cd^2)(16a^4b^2 - 12a^2bcd^2 + 9c^2d^4)$;
$\frac{1}{64}(ab + 4c^2)(ab - 4c^2)(a^2b^2 - 4abc^2 + 16c^4)$;
$(a^2b^2 + 4abc^2 + 16c^4)$; $7abc(2b - c)(4b^2 + 2bc + c^2)$;
$(4a^4 + 9b^4)(16a^8 - 36a^4b^4 + 81b^8)$;
$\frac{abcd}{216}(2ab + 5cd)(4a^2b^2 - 10abcd + 25c^2d^2)$;* $(a + b + c + 1)$ &c.;
$(2a^2 + \frac{2}{a^2})$, &c.; $\frac{2}{b^8c^3}(a^6 + 3b^6)$, &c.; $\frac{x(2a + x)}{a(a + x)}$, &c.;
$\frac{1}{(x + 1)^6}$, &c.; $(3x + 4y)$, &c.; $(a + 1)$, &c.; $(1 + x)$, &c.;
$(3x + 2)$, &c.; $(x^2 - 3xy + y^2)$, &c.; $(x^4 + 12x^2y^2 + 16y^4)$, &c.;
$(a + x - x^2 - \frac{x}{a} - \frac{x^2}{a} - \frac{1}{a^2})$, &c.; $\frac{x + y}{6}$, &c.;
$\frac{a^2b^2c^2}{6}(2 - 3abc)$, &c.

* Below this point all the factors are not given.

### XXXVI.—P. 96.

$(x + 1)(x^4 - x^3 + x^2 - x + 1)$ or $(x - 1)(x^4 + x^3 + x^2 + x + 1)$;
$(x \pm 1)$, &c.; $(a \pm 2)$, &c.; $x^5y^5(x \pm y)$, &c.; $(a^2 + ab + b^2)$, &c.;
$2a$, $2b$, &c.; $\frac{5}{6}(x + y)$, &c.; $\frac{(a^3 + abc + b^3)}{a^6b^6}$, &c.;
$\frac{c^5}{a^{10}b^{10}}(a + b)$, &c.; $\frac{3}{16}(x - 1)$, &c.; $(x^2 + y^2)$, &c.; $\frac{x^4 + 1}{x^2}$, &c.;
$\frac{2(x^2 + 1)}{(x - 1)^9(x + 1)^9}$, &c.; $\frac{x^{14}}{(1 - x)^7(1 + x)^7}$.

### XXXVII.—P. 97.

$a^4(25a^2 - 50ab + 2b^2)(25a^2 + 50ab + 2b^2)$; $(x^2 - 4x + 8)$, &c.;
$(9a^2 - 6ab + 2b^2)$, &c.; $(\frac{x^2}{y^2} - 2\frac{x}{y} + 2)$, &c.;
$(a^2 + 2ab + b^2 - 2a - 2b + 2)$, &c.; $(a^2 - 2ab + 5b^2)$, &c.;
$(x + y)^8(x^4 - 2x^2y^2 + 2y^4)$, &c.; $(a^2 + b^2 + 5c^2 - 2ac - 2bc + 2ab)$,
&c.; $(16a^4 - 24a^3b + 50a^2b^2 - 24ab^3 + 16b^4)$, &c.;
$\frac{1}{9}(\frac{5x^4}{y^2} + 14xy + 17\frac{y^4}{x^2})$; $(5x^4 + 10x^3 + 19x^2 + 14x + 17)$, &c.;
$\{x^4(25a^2 - 10ab + 2b^2) + x^2(50a^2 - 4b^2) + 25a^2 + 10ab + 2b^2\}$, &c.

### XXXVIII.—P. 98.

$(3ax - \frac{by}{2})^2$; $(ab + \frac{1}{2})^2$; $(\frac{x}{2} + \frac{1}{2})^2$; $(\frac{x}{3} + 6y)^2$; $(\frac{a}{2} - \frac{b}{3})^2$;

$(\frac{x^2}{3}+\frac{1}{7})^2$; $(a-b+\frac{1}{6})^2$; $(7ax-\frac{y}{7})^2$; $(\frac{3}{2}x^2-\frac{2}{3}y^2)^2$;
$(a+3b-1)^2$; $(x+6)^2$; $(x+\frac{3}{2}y+\frac{5}{2})^2$; $(\frac{4}{5}x^3-\frac{5}{4}y^3)^2$;
$(3x^5-\frac{1}{4}x^2y^3)^2$; $(\frac{7}{3}ab+\frac{2}{5}bc)^2$; $(a+b+\frac{1}{2})^2$; $(\frac{x}{9}-\frac{y}{8})^2$;
$\left(\frac{a+b}{2}-\frac{1}{3}\right)^2$; $\left(\frac{a^2+1}{9}-\frac{c^2}{4}\right)^2$.

### XXXIX.—P. 99.

$(a^3b^4-a^4b^3)$, &c.; $(ab^2c^3+2d^4)$, &c.; $(y^4+3ax^3)$, &c.; $(a^3-2bc^2)$, &c.;
$(a^3d^3+3b^3c^3)$, &c.; $(x^3y^2-4z^2)$, &c.; $(ab-13c)$, &c.;
$7(x^3+2y^3)$, &c.; $(x^4+1)$, &c.; $\left(\frac{x^2}{y^2}+5\right)$, &c.; $\left(\frac{a}{b}+1\right)$, &c.;
$(a-3)$, &c.; $2x$, &c.; $(8xy-3x^2-3y^2)$, &c.

### XL.—P. 101.

One only of the factors is given.

$x+4$; $3x-7$; $x+4$; $6x-5$; $8x-3$; $2x^2-3$; $10x^2-9$; $3a-2b$;
$3x-13y$; $10x^2-y^2$; $x^2+3y^2$; $a^3+8b$; $2a^4-9b^4$;
$8a^2b^2+3c^2d^2$; $a-2b$; $a+b$; $3ax+2by$; $x^6-3y^6$; $x^2y-yz^2$;
$m-5n$; ; $m^2n^2-6x^2y^2$; $x-9$; $7x-8y$; $49x^3+y^3$;
$3ab+2cd$; $4a^2+3b^2$; $11x^3-1$; $x-3$; $12x-1$; $24a^2+b^2$;
$a+3b$; $x-3y$.

See Table of Errata for the last three lines beginning $a^4+\frac{577}{24}a^2b^2$, &c.

### XLIV.—P. 104.

$x^2y+y^2z-x^2z$; $2xy-3yz+4xz$; $2by-4cz+ax$; $2a-b-\frac{1}{2}$;
$\frac{2a}{3}-\frac{b}{2}-\frac{3c}{2}$; $4x^3+4x^2y-\frac{1}{2}$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$; $2a^3-2ab+b^2$;
$\frac{a^2}{b^2}+\frac{a}{b}-1$; $\frac{a^3}{b^3}-\frac{a^2}{b^2}-\frac{a}{b}$; $a+3+\frac{1}{a}$; $a+b-c-d$;
$\frac{x^3}{y^3}+\frac{x^2}{2y^2}-\frac{x}{2y}$; $\frac{a^3}{b^2c}-\frac{b^3}{ac^2}-\frac{c^3}{a^2b}$.

### XLV.—P. 105.

$a^2+4b^2$, &c.; $a^2+x^2$, &c.; $x-2y$, &c.; $a^2+b^2$, &c.; $x-ab$, &c.;
$2a-3b$, &c.; $p-10q$, &c.; $3cd-4zv$, &c.; $p^4+4q^4$, &c.;
$\frac{b}{3}-\frac{c}{4}$, &c.

### XLVI.—P. 105.

$x + y + 2z$, &c.; $3x^2 - y^2 - 1$, &c.; $7ab - 3ac - 9bc$, &c.;
$\frac{x}{2} - \frac{y}{3} - \frac{z}{4}$, &c.; $\frac{x^2}{2y^2} - \frac{x}{y} + 2$; $\frac{a}{b} + \frac{c}{d} - \frac{e}{f}$; $3a^2 + b^2$, &c.

### XLVII.—P. 106.

$a - b + 5c$, &c.; $x + 2y + z$, &c.; $2xy - 2xz + 5yz$, &c.;
$3a - b + c$, &c.; $2x + y - z$, &c.;
$\frac{1}{ax} - \frac{2}{by} - \frac{5}{cz}$, &c.; $x + y$, &c.; $\frac{2x}{a} - \frac{z}{2c}$, &c.

### XLVIII.—P. 107.

$a^2 - 2b^2 + c^2$; $a^2 + 2b^2 - c^2$, &c.; $x + y + z$; $x + y + 1$;
$2p^2 + 3q^2 - 4r^2$; $\left(\frac{1}{x^2} - \frac{2}{y^2} - \frac{3}{z^2} - \frac{\sqrt{12}}{xz}\right)$; $3a^2x^2 - 4b^2y^2 - 5c^2z^2$;
$\frac{p}{2bc} + \frac{q}{2ac} + \frac{r}{ab}$.

### L. P. 115.

(4.) $a^{13}$; $a^{\frac{(m+n)2}{mn}}$; $a^{mn+mp+np}$; $\left(\frac{x}{y}\right)^{np+mq}$; $a^{\frac{227}{60}}$; $7x^2y^{\frac{3}{2}}z$; $\frac{a^2}{b^{\frac{5}{3}}}$;
$\frac{1}{2x^4y^2z^{\frac{6}{5}}}$; $\frac{x^{\frac{1}{6}}}{4}$; $(ax)^{\frac{2}{3}}$; $\left(\frac{3}{2}\right)^{\frac{1}{2}} \cdot \left(\frac{a}{x}\right)^{\frac{2}{3}}$; $\frac{2}{3}\left(\frac{y}{x}\right)^{\frac{1}{6}}$; $x^{\frac{2}{p}}$; $(6x^2)^{\frac{1}{3}}$;
$\frac{1}{b^{\frac{4}{3}}}$; $\frac{x^{\frac{mn}{q}}}{y^{\frac{mn}{p}}}$; $\frac{a^{\frac{n}{2}} \cdot b^m \cdot c^{\frac{m+3}{2}}}{x^{\frac{m-2}{2}} \cdot y^{\frac{m-3}{2}}}$.

(5.) $\frac{x^3}{y^2}$; $\frac{x^4}{y}$; $\frac{256}{81}\,\frac{a^2}{b}$; $-\frac{a}{b^9x^3}$; $x^7$.

(7.) $a - c^2d^{\frac{2}{3}}$; $a^{\frac{3}{4}} - a^{\frac{7}{10}} - a^{\frac{7}{12}} + a^{\frac{8}{15}} + a^{\frac{1}{2}} - a^{\frac{9}{20}}$; $a^m - b^m$;
$a^2 + 6ac^{\frac{1}{3}} + 9c^{\frac{2}{3}} - 4b$; $p^{\frac{1}{2}}q + p^{\frac{1}{4}}q^{\frac{2}{3}} - p^{\frac{1}{4}}q^{\frac{4}{3}} - q$; $x^3 - 729y^2$;
$(a^2 - x^2)^{\frac{m+n}{mn}}$; $x + x^{-\frac{1}{3}} - 1 + x^{-\frac{1}{3}} + x^{-1}$; ———.
$a^{4n} - a^{2n} + a^{-2n} - a^{-4n}$; $(abc)^{\frac{7}{6}}\,(a^{\frac{1}{3}} + c^{\frac{1}{3}} + 2a^{\frac{1}{6}}c^{\frac{1}{6}} - b^{\frac{1}{3}})$.

(8.) $x^{\frac{8}{3}} + x^{\frac{7}{3}} + x^2 + x^{\frac{5}{3}} + x^{\frac{4}{3}} + x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$; $x + 1 + x^{-1}$;

$x^{-\frac{7}{6}} - x^{-4} + x^{-\frac{25}{6}}$, &c.; $x - x^{\frac{2}{3}} + x^{\frac{1}{3}}$, &c.;

$x^{\frac{7}{2}} + 2x^3y^{\frac{1}{2}} + 4x^{\frac{5}{2}}y + 8x^2y^{\frac{3}{2}} + 16x^{\frac{3}{2}}y^2 + 32xy^{\frac{5}{2}} + 64x^{\frac{1}{2}}y^3 + 128y^{\frac{7}{2}}$.

(9.) $3a^{\frac{3}{2}} - b$; $\frac{1}{2}a^2 \cdot 2b^{\frac{1}{2}}$; $\frac{2a^m}{b^n} - \frac{3b^n}{4a^m}$; $\frac{a^2b^{\frac{1}{2}}}{2} - \frac{a}{b^{\frac{1}{2}}} + \frac{1}{2b^{\frac{3}{2}}}$;

$\frac{2}{5}a^{2m}b^{\frac{1}{m}} - 5a^{-m}b^{\frac{2}{m}}$; $x^{\frac{1}{m}}y^m - \frac{x^2}{2y^m}$; $x^{\frac{3}{p}} + x^{\frac{1}{p}} + x^{-\frac{1}{p}}$.

(10.) $x + \frac{1}{x}$; $\frac{1}{2a^2} + 1$; $\frac{x^2y^{\frac{1}{2}}}{2} - \frac{x}{3y^{\frac{1}{2}}}$; $2a^m + \frac{a^{-\frac{m}{2}}}{2}$; $x^2 - x^{\frac{4}{3}} + x$;

$x^{\frac{1}{3}} + a^{\frac{1}{3}}$; $(5x^{-\frac{1}{3}} - x^{-\frac{13}{9}})y^{\frac{2}{15}}$; $x^{-\frac{23}{12}}$, &c.;

$$\frac{1}{(3x^2y^{-3} + 4x^{-1}y^2)^3} \cdot \frac{1}{\sqrt[3]{\frac{x}{y}} - \sqrt[3]{\frac{y}{x}}} \cdot \frac{1}{\left(\sqrt[3]{\frac{x}{y}} + \sqrt[3]{\frac{y}{x}}\right)^2}.$$

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### LI.—P. 134.

3. $\left(2^{\frac{2}{3}}x^{\frac{2}{3}}yz^{\frac{4}{3}}\right)^3$; $\left(\frac{m}{n}\right)^2$; $\left(\frac{m^{\frac{1}{3}}}{n^{\frac{1}{12}}}\right)^4$; $\sqrt[3]{a^2 + 2ab + b^2}$, &c.

4. $(a^2b^3)^m$ and $\left(a^{2m^2}b^{3m^2}\right)^{\frac{1}{m}}$; $\left(\frac{x^{\frac{2}{m}}}{y^{\frac{2}{m}}}\right)^m$ and $\left(\frac{x^{2m}}{y^{2m}}\right)^{\frac{1}{m}}$, &c.

(8.) $3a^2b^2 . a^{\frac{1}{3}}$; $\frac{x}{y} . z^{\frac{1}{2}}$; $2ay^2 . \left(\frac{ay}{9}\right)^{\frac{1}{4}}$; $\frac{c}{a^2b} . c^{\frac{1}{4}}$ &c.

(9.) $\frac{\sqrt{x^2 + 1} + 1}{x}$; $\frac{\sqrt{a}(\sqrt{a} - 1)}{a - 1}$; $\frac{3\sqrt{xy}(x - \sqrt{y})}{x^2 - y}$; $\frac{x + \sqrt{x^2 - y^2}}{y}$;

$\frac{(\sqrt{2 + x} - \sqrt{x})^4}{4}$; $1 + x + \sqrt{2x + x^2}$; $\frac{(a - \sqrt{2ax - x^2})^2}{(a - x)^2}$;

$\frac{(\sqrt{a^2 + ab} - a)(a + b)}{b}$; $\frac{(1 + \sqrt{1 + a})(1 - \sqrt{1 - a})}{a}$;

$\frac{2x^2 + y^2 + 2x\sqrt{x^2 + y^2}}{y^2}$, &c.

(10.) .8684; 1342; 1.1153; 2.932; 3.144; .2426; 5.8284.

(11.) The Rational Denominators only are given.
$1 + x^2 + y^2 - 2x - 2y - 2xy$; $x^2 + xy + y^2$; $2ax$; 4; 3816; 11; 5; 598; 14; 2; 239; $2^{20}.x^{15} - 3^{20}.y^4$; $a^2 - 2ax + x^2 - a - x$; $x$; 116; 503.

(12.) $\frac{\sqrt{6}}{3}$; $2\sqrt{a^2-1}$; $\frac{4a\sqrt{a^2-b^2}}{b^2}$; $\frac{\sqrt{a^4-16}}{2}$; 0; 1.

(13.) $\frac{\sqrt{3}+1}{\sqrt{2}}$; $3-\sqrt{2}$; $3-2\sqrt{2}$; $\sqrt{11}+2\sqrt{13}$; $\frac{1+2\sqrt{2}}{2}$;

$\sqrt{7}-1$; $\frac{4-3\sqrt{3}}{6}$; $\frac{20-6\sqrt{11}}{3\sqrt{11}}$; $2\sqrt{2}+2\sqrt{5}$; $3^{\frac{1}{4}}(5-2\sqrt{3})$;

$7^{\frac{1}{4}}(\sqrt{6}+\sqrt{5})$; $\frac{\sqrt{7}+4}{(14)^{\frac{1}{4}}}$; $\frac{\sqrt{7}-2}{(11)^{\frac{1}{4}}}$; $\sqrt{a^2+b^2}(a+b)$;

$a+1-\sqrt{a^2-1}$; $a+b+c+\sqrt{a^2+b^2+c^2}$.

(14.) $\sqrt{2}-1$; $\sqrt{7}+1$; $\sqrt{2}+\frac{1}{\sqrt{3}}$; $(5-2\sqrt{2})^{\frac{1}{2}}$; $\frac{3^{\frac{1}{8}}}{2^{\frac{1}{4}}}(\sqrt{3}+1)$.

## LII.—P. 139.

$x = \pm 2$; $x = \pm 5$; $x = \pm 1$; $x = \pm\sqrt{3}$; $x = \pm\sqrt{-2}$; $x = \pm 2$;

$x = \pm 4$; $x = \pm\frac{12}{7}$; $x = \pm 3$.

The last three are adfected quadratics.

## LIV.—P. 141.

(1.) $x = 7$ or $1$; $x = 1$ or $-\frac{3}{2}$; $x = -\frac{3}{4}$ or $-1$; $x = 12$ or $-1$;

$x = -5$ or $\frac{1}{4}$; $x = 9$; $x = -\frac{1}{4}$; $x = 5$ or $-\frac{17}{4}$; $x = 1$ or $3$;

$x = 3$ or $5\frac{1}{4}$; $x = -\frac{41}{7}$ or $-\frac{31}{11}$; $x = \frac{7}{2}$ or $\frac{21}{8}$; $x = \frac{1}{3}$ or $-\frac{1}{4}$;

$x = 2$ or $-\frac{13}{6}$.

### P. 142.

(2.) $x = 1$ or $-7$; $x = 0$ or $\frac{17}{9}$; $x = \frac{5}{2}$ or $-4$; $x = -3$ or $\frac{1}{2}$;

$x = -\frac{19}{18}$ or $-\frac{16}{3}$; $x = \frac{-49 \pm \sqrt{-45224}}{50}$; $x = 13$ or $\frac{324}{13}$

$x = -\frac{1}{3}\left\{a+b+d \pm \sqrt{\overline{a+b+d}|^2 - 9(ab+ad+bd)}\right\}$.

### P. 143.

(3.) $x = \frac{2}{3}$ or $-2$; $x = \frac{-3\pm\sqrt{5}}{2}$; $x = \pm 2$; $x = 8$; $x = \pm 3$;

$x = 6$ or $-\frac{4}{5}$; $x =$ [illegible] or $\frac{[illegible]}{3}$; $x = 30$ or $13$; $x = -4$; $x = 7$;

$x = \pm\frac{1}{2}$; $x = 3\frac{1}{4}$; $x = 3$.

### P. 144.

$x = 4$ or $-1\frac{2}{3}$; $x = \frac{115}{36}$ or $\frac{5}{4}$; $x = \pm\frac{1}{2}$; $x = \pm 3$; $x = 3$ or $-\frac{25}{8}$;

$x = \pm 9$; $x = 6$ or $3$; $x = 9 \pm \sqrt{\frac{361}{5}}$; $x = 6$ or $\frac{19}{3}$;

————; $x = 4$ or $-\frac{5}{3}$; $x = 1$ or $\frac{29}{13}$; $x = 3$ or $-\frac{2}{3}$;

$x = 2$ or $-4\frac{1}{6}$; $x = 5 \pm \sqrt{22}$; $x = 4$ or $-9$; $x = 1$ or $-2$;

$x = (2a + 1)$, or $-1$.

### P. 145.

$x = \frac{b}{2}$ or $-\frac{2a+3b}{2}$; $x = \frac{d}{c}$ or $-\frac{b}{a}$; $x = \frac{1}{b} \pm \frac{1}{a}$; $x = \frac{3a}{4}$ or $\frac{a}{2}$;

$x = \frac{ac}{b}$ or $\frac{bc}{a}$; $x = 2ab - ad$; $x = -a \pm \frac{ab}{\sqrt{b^2 - c^2}}$.

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### LV.—P. 146.

$x = \pm\sqrt{-3}$; $x = 3$ or $\sqrt[3]{-41}$; $x = 3\sqrt[3]{2}, -2\sqrt[3]{2}$;

$x^2 = \frac{6(\sqrt{40} - 1)}{3(\sqrt{10} - 2)}$; $x = \pm 2$; $x = \pm\frac{3}{2}$; $x = \pm\frac{2}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}$;

$x = 3, -1$; $x = \pm\sqrt{2}, \pm\sqrt{\frac{3}{5}}$; $x = 1, 2, \pm\sqrt{2}$;

$x = \pm\sqrt{\frac{a-1}{a+1}}, \pm\sqrt{\frac{a}{a-2}}$; $x = \pm\sqrt{1 \pm \sqrt{2}}$;

$x = 1, 2^{\frac{2(p+q)}{p}}$.

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### LVI.—P. 150.

(1.) $x = 1, \frac{\sqrt{3}}{2}(\sqrt{5} - 1)$; $x^2 = \frac{21}{8} \pm \frac{\sqrt{45}}{4}$; $x = 4, -1$; $x = 4, 2,$

$\frac{-7 \pm \sqrt{17}}{2}$; $x = 3, -\frac{1}{2}$; $x = \pm 1, \pm \sqrt[6]{-5}$; $x = \pm 1, \pm \sqrt[6]{-4}$;
$x^2 (17 \pm \sqrt{33}) = 3x + 26$; $x = -1$; $x = 2, -\frac{50}{391}$;
$x = 0, \sqrt[6]{\frac{27}{28}}$; $x = 1 \pm \sqrt{3}$; $x = \pm 5, \pm 4\sqrt{2}$; $x = 1, \frac{1}{81}$;
$x = 0, \frac{35}{37} a$; $x = 1, -2$; $x = \frac{1}{2}\left(-3 \pm \sqrt{7}\right)$; $x = 4, -2,$
$-1 \pm \sqrt{7}$; $x = \frac{1}{2}\left(7 \pm \sqrt{13}\right)$; $x = -1, \frac{3}{2}$; $x = \pm \frac{\sqrt{3}}{2}$,
$\frac{1 \pm \sqrt{7}}{4}$.

P. 151.

(2.) $x = -2, \frac{1}{2}\left(1 \pm \sqrt{-39}\right)$; $x = -2, -2 \pm \sqrt{3}$; $x = 2, 2 \pm \sqrt{3}$;
$x = 1, \frac{1}{4}\left(-1 \pm \sqrt{-7}\right)$; $x = 1, \frac{1}{2}\left(-3 \pm \sqrt{-7}\right)$; $x = c,$
$\frac{1-c}{c}, -\frac{1}{1-c}$; $x = 2, -1$; $x = -1, \frac{1}{4}\left(5 \pm \sqrt{-15}\right)$;
$x = a, 4, 9.$

The second member of this should be $x^{\frac{1}{2}}\left(a^{\frac{1}{2}} + 6\right)$.

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## LVII.—Pp. 154, 155.

(1.) $x = 6$; $x = 2$; $x = \frac{1}{4}$ or $\frac{17}{4}$; $x = \frac{(p-q)^2}{2p-q}$; $x = 7$ or $1$;
$x = \pm 5\sqrt{2}$; $x = 6$; $x = \frac{1}{4}$ or $\frac{1}{9}$; $x^2 = \frac{27c^2 + 4a^2}{12}$; $x = 5$;
$x = \frac{a(b+c)}{b-c}$; $x = \pm \frac{3b}{\sqrt{(9b-a)(a-b)}}$;
$x = -\frac{3}{2} \pm \sqrt{\frac{- \pm \sqrt{-55}}{8}}$; $x = 1$ or $-\frac{1}{2}$;
$x = 16$ or $\frac{1}{16}$; $x = 27$ or $12$; $x = 8$; $x = \frac{9}{16}$ or $\frac{16}{9}$;
$x^2 = \frac{-1 \pm \sqrt{-3}}{2}$; $x = \frac{2c}{a}\left(\frac{c}{b} - 1\right)$; $x = \pm \frac{18}{\sqrt{2}}$.

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## LIX.—P. 163.

$$\left.\begin{array}{l} x = 3 \text{ or } -\frac{5}{2} \\ y = 18 \text{ or } \frac{25}{2} \end{array}\right\} \qquad \left.\begin{array}{l} x = 10 \text{ or } \frac{7}{2} \\ y = 7 \text{ or } 20 \end{array}\right\} \qquad \left.\begin{array}{l} x = 3 \\ y = 2 \end{array}\right\}$$

$$\left.\begin{array}{l} x = 25 \text{ or } -10 \\ y = 10 \text{ or } -25 \end{array}\right\} \qquad \left.\begin{array}{l} x = 30 \\ y = 20 \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm\frac{17\sqrt{3}}{2} \\ y = \pm\frac{\sqrt{3}}{2} \end{array}\right\}$$

## P. 164.

$$\left.\begin{array}{l} x = 2 \text{ or } 5 \\ y = 6 \text{ or } 3 \end{array}\right\} \qquad \left.\begin{array}{l} x = 4 \\ y = 2 \end{array}\right\} \qquad \left.\begin{array}{l} x = 4 \\ y = 3 \end{array}\right\}$$

$$\left. y = \frac{-5 \pm \sqrt{-775271}}{48} \right\} \qquad \left.\begin{array}{l} x = \pm 12 \\ y = \pm 16 \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm 10 \\ y = \pm 6 \end{array}\right\}$$

$$\left. y^2 = \frac{8}{41 \pm \sqrt{1649}} \right\} \qquad \left.\begin{array}{l} x = \pm\frac{15}{\sqrt{2}} \\ y = \pm\frac{1}{\sqrt{2}} \end{array}\right\} \qquad \left. y^2 = \frac{504}{4 \pm \sqrt{51}} \right\}$$

$$\left.\begin{array}{l} x = \pm 5 \\ y = \pm 3 \end{array}\right\} \qquad \left. x = \frac{-5 \pm \sqrt{8755}}{9} \right\} \qquad \left.\begin{array}{l} x = \pm 7 \\ y = \pm 4 \end{array}\right\}$$

$$\left.\begin{array}{l} x = 8 \\ y = 5 \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm 40 \\ y = \pm 1 \end{array}\right\} \qquad \left. x = \frac{b(a+1)}{\sqrt{2}\sqrt{a^2+1}} \right\}$$

Omit the equations $x - y : x + y :: 1 : 15$;

$$\left.\begin{array}{l} x = \frac{1 \pm \sqrt{2}}{4} \\ y = \frac{1}{4} \end{array}\right\}$$

$$\left.\begin{array}{l} x = \frac{-5 \pm \sqrt{142}}{3} \\ y = 3 \end{array}\right\} \qquad \left.\begin{array}{l} x = 16 \text{ or } 8 \\ y = 8 \text{ or } 16 \end{array}\right\}$$

$$\left.\begin{array}{l} x = -8 \\ y = 12 \end{array}\right\} \qquad \left.\begin{array}{l} x = 11 \\ y = 10 \end{array}\right\} \qquad \left.\begin{array}{l} x = 22 \text{ or } \frac{26}{43} \\ y = 11 \text{ or } -\frac{52}{43} \end{array}\right\}$$

$$\left.\begin{array}{l} x = 5 \\ \;\;\, = 2 \end{array}\right\} \qquad \left.\begin{array}{l} x = 7 \\ y = 3 \end{array}\right\} \qquad \left.\begin{array}{l} x = 4 \\ y = 2 \end{array}\right\}$$

$$\left.\begin{array}{l} x = \pm\sqrt{3} \\ y = \pm\sqrt{2} \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm\dfrac{2}{\sqrt{5}} \\ y = \pm\dfrac{1}{\sqrt{5}} \end{array}\right\} \qquad \left.\begin{array}{l} x = \dfrac{27}{7} \text{ or } \dfrac{64}{7} \\ y = \dfrac{4}{3} \text{ or } \dfrac{3}{4} \end{array}\right\}$$

$$x = \left(\frac{1}{2} \pm \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{1}{2} \pm \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}}\Bigg\} \qquad \left.\begin{array}{l} x = \dfrac{1}{\sqrt{3}} \text{ or } 1 \\ y = \dfrac{4}{\sqrt{3}} \text{ or } 2 \end{array}\right\}$$

$$x = \frac{\sqrt{2b-a} + \sqrt{a}}{2\sqrt[6]{2b-a}}\Bigg\} \qquad \left.\begin{array}{l} x = 2 \\ y = 2 \end{array}\right\} \qquad \left.\begin{array}{l} x = 4 \\ y = 8 \end{array}\right\}$$

P. 165.

$$\left.\begin{array}{l} x = 8 \text{ or } 2 \\ y = 2 \text{ or } 8 \end{array}\right\} \qquad \left.\begin{array}{l} x = 1 \text{ or } 9 \\ y = 9 \text{ or } 1 \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm 8\sqrt{2} \\ y = \pm\sqrt{2} \end{array}\right\} \qquad \left.\begin{array}{l} x = 4 \text{ or } 2 \\ y = 2 \text{ or } 4 \end{array}\right\}$$

$$y = \frac{-15 \pm \sqrt{109}}{4}\Bigg\} \qquad \left.\begin{array}{l} x = 4 \\ y = 9 \end{array}\right\} \qquad \left.\begin{array}{l} x = \pm\dfrac{9}{2} \\ y = \pm\dfrac{7}{2} \end{array}\right\} \qquad \left.\begin{array}{l} x = 8 \text{ or } 1 \\ y = 1 \text{ or } 8 \end{array}\right\}$$

$$x = a.\frac{-11 \pm \sqrt{153}}{16}\Bigg\} \qquad x^{\frac{1}{p}} = \frac{1}{2}\left\{\sqrt{a^{\frac{2}{p}} - 4c^{\frac{2}{p}}} + a^{\frac{1}{p}}\right\}$$

$$y = z = \frac{3 \pm \sqrt{45}}{2}\Bigg\} \qquad \left.\begin{array}{l} x = \dfrac{c^2 - a^2}{a} \\ y = a \\ z = a \end{array}\right\}$$

## LX.—P. 170.

(1.) 16 and 4. (2.) 13 and 2. (3.) 4. (4.) 10 and 4. (5.) 6 and 6. (6.) $\dfrac{a + \sqrt{a^2 - b^2}}{2}$ and $\dfrac{a - \sqrt{a^2 - b^2}}{2}$. (7.) $16\frac{2}{3}$ or 10. (8.) 400 and 1600. (9.) 4.14 and 6.14 feet. (10.) 10, 12, and 14 feet. (11.) 1225. (12.) 576. (13.) 24. (14.) 223. (15.) 3 and 4. (16.) $8\frac{2}{11}$ and $11\frac{1}{4}$ nearly. (17.) 20. (18.) $\frac{1}{5}$. (19.) 30 miles. (20.) 10 and 15. (21.) 3, 4, and 5 the least. (22.) 15. (23.) $482\frac{1}{2}$ nearly. (24.) 1240 the least.

## LXI.—P. 174.

(1.) 2 and 3. (2.) 3 and − 2, (3.) 4 and 2 ; $20\sqrt{2}$ and $7\sqrt{2}$. (5.)

2700 and 9. (6.) 114. (7.) 164½ and 135½. (8.) 6, 5, and 4 the greatest. (9.) 54 and 5 nearly for the rectangle. (10.) 223. (11.) 5 days. (12.) 4536. (13.) 200, 40 and 32. (15.) $\dfrac{a \pm \sqrt{a^2 - 2b}}{2}$, 5 sov. 20 cr. (16.) 9, 6. (17.) 40 at 5*s*., or 12½ at 1*s*. 4*d*. (18.) 4½ each. (19.) 1300, 1200, 500. (20.) 20. (21.) 12, 6, 4. (22.) 361. (24.) 2, 5, 11.

### LXII.—P. 181.

(1.) $\dfrac{2}{5}$; $\dfrac{2}{3}$; 4; $\dfrac{3}{4}$; $\dfrac{2}{3}$; 2; 2; $\dfrac{1}{b}$; $\dfrac{a}{b}$; $\dfrac{a+b}{a}$; $a$; $\dfrac{x-y}{x+y}$; $\dfrac{x-y}{x+y}$; $\dfrac{xy}{x^2+y^2}$; $\dfrac{1}{a^2-b^2}$; $\dfrac{xy\,(a^2-b^2)}{ab\,(x^2-y^2)}$; $\dfrac{1}{\sqrt{x}-\sqrt{a}}$; $\dfrac{1}{x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}}$; $\dfrac{x-y}{x+y}$; 1; $3+2\sqrt{2}$; $\dfrac{1}{2}\sqrt[6]{\dfrac{81}{2}}$; $\sqrt[4]{\dfrac{8}{9}}$. (2.)

(3.) $1:2$; $2:5$; $1:3$; $a:c$; $a^2-4ax+3x^2 : a^2+4ax+3x^2$.

(4.) $4:9$; $1:100$; $4:1$; $9:4$; $1:4$; $a^2:c^2$; $a^2c^2:b^4$; $(a+x)^2:(a-x)^2$; $(a+1)^2:(a-1)^2$; $a+x:a-x$; $a^{\frac{1}{2}}+b^{\frac{1}{2}}:a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

(5.) $125:1$; $125:8$; $(a-1)^3:a^3$; $(xy)^{\frac{3}{2}}:(x+y)^{\frac{3}{2}}$; $a:b$; $(a-x)^6:(a+x)^9$; $4:3$; $1:2$; $3:2$; $ax:a+x$; $(x^n+a^m)^{\frac{1}{2}}:(x^n-a^m)^{\frac{1}{2}}$; $3:2$; $2:1$; $a+x:1$; $(a+x)^2:(a-x)^2$.

(6.) 1 : 32, &c. (7.) 7 and 21. (8.) 12700 at the beginning of the second year.

### LXIII.—P. 191.

(1.) $x:4::a:y$; $x^2:36::1:9y^2$; $a-b:x-1::x+1:a+b$; $a-1:b+1::b^2-b+1:a^2+a+1$; $x+1:y-1::y-4:x+4$.

(2.) *Inversion.* $1:a::2:b$; $a:b::y:x$.
*Comp.* $2x-a:x-a::y:y-b$; $a^2:x^2::b^2:y^2$.
Div. $x:1-x::y:3-y$; $a^2:x^2::b:y$.

(7.) 12 and 30. (8.) $x = \dfrac{ad-bc}{b+c-a-d}$.

(13.) 66666⅔; 15; $\left(1-\dfrac{1}{n}\right)^m$ denotes the part left.

## LXIV.—P. 198.

1.) $y = 2x$. (2.) $y^2 = \frac{3x}{2}$. (4.) $3z = 4(x + y)$. (6.) $\frac{x}{a} + \frac{y}{b} = 1$. (7.) $c(x^2 + y^2) = a^2z^2$. (8.) $x = 2$. 18. $x = 24$. (20.) 16 : 1. (21.) 1 ft. (22.) 36 : 49. (23.) 32.2 and 2.73. (24.) 2.009; 5 = $16\frac{1}{10}b^2$. (25.) $\sqrt[3]{91}$. (26.) 2 and 113.0976. (27.) $20\frac{5}{9}$.

## LXV.—P. 206.

1.) 41. (2.) $2a + \overline{n-1}c$; $36a + 153c$. (3.) $\frac{b-a}{4}$ and $\frac{11a - 7b}{4}$. (4.) $a$; $\frac{b-a}{13}$. (5.) 6; 25. (6.) 2; 3; 4. (7.) 3; 4; 5. (8.) $a + c = 2b$ and $a - b = b - c = c - d$. (10.) 45; 60; 75. (13.) $b^2 + c^2 + (n+1)bc$. (14.) 10 and $\frac{n}{n+1}$. (15.) 2; 6; 10; 14; &c.; and $p$; $3p$; $5p$; &c.

## LXVI.—P. 211.

1.) $2^{20}$; $5 \times 4^{15}$; $(3x)^n$; $(-1)^{n-1}\frac{x^{n-1}}{2^{n-2}}$; $(a-x)^{2n-1}$; $\frac{x^{n-3}}{a^{n-2}}$; $\frac{3}{(-2)^{n-2}}$.

(2.) $\frac{7}{2}(3^{19} - 1)$; 1640; $\frac{1}{4}(3^8 - 1)$; $8\left\{1 - \left.\overline{\frac{1}{2}}\right|^{16}\right\}$; $8\left\{1 - \left.\overline{\frac{1}{2}}\right|^{n}\right\}$.

(2.) $\frac{3^{30} - 2^{30}}{3^{28}}$; $18\left\{1 - \left.\overline{\frac{2}{3}}\right|^{3n}\right\}$; $\frac{(x-1)^2}{x}\left\{\left.\overline{\frac{x}{x-1}}\right|^{n} - 1\right\}$;

$(a^2 + ax + x^2)\dfrac{\left.\overline{a-x}\right|^{n+1} - 1}{a - x - 1}$; $(a+x)^3\dfrac{1 - \left.\overline{\frac{a-x}{a+x}}\right|^{n}}{2x}$;

$(\sqrt{2} - 1)\frac{(\sqrt{2}+1)^n - 1}{\sqrt{2}}$. (3.) 4, 8, 16, &c.; $r = \sqrt[3]{3}$; $r = (2)^{\frac{13}{17}}$; $r = \frac{1}{a+b}$; $r = \frac{1}{\sqrt[17]{a^5x^3}}$. (4.) $r = 2^{\frac{10}{9}}$ and $2^{\frac{20}{3}}$.

(5.) 1st term = $\frac{4^{22}}{5^{19}}$ and 100th term = $\frac{5^{80}}{4^{77}}$. (6.) $\left(\frac{b^{m-1}}{a^{n-1}}\right)^{\frac{1}{m-n}}$.

(7.) $r = \frac{\sqrt{5} - 1}{2}$. (8.) $r = 3^{\frac{2}{x}}$. (9.) 8 per cent. (10.) 5 per cent. (13.) $\frac{9^{n}-1}{4}$; $\frac{1}{9}\left\{\overline{1.5}|^{n}-1\right\}$. (15.) $340 \mp 240\sqrt{2}$.

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## LXVII.—P. 214.

(1.) $2$; $\frac{9}{2}$; $\frac{1}{8}$; $12$; $16$; $\frac{ar}{1-ar}$. (2.) $\frac{2}{3}$; $\frac{4}{5}$; $\frac{1}{4}\left(1-\frac{1}{2n}\right)$; $\frac{1}{4}$.
(3.) $\frac{3^{201}}{5\times 8^{198}}$. (4.) $\frac{1}{3}$; $\frac{5}{11}$; $\frac{1}{150}$; $\frac{41}{133}$; $\frac{3}{7}$; $\frac{7}{1100}$;
Diff. $=\frac{7}{1100000000}$. (5.) Nine places. (6.) $\frac{3}{8}\left\{1-\overline{\frac{2}{3}}\Big|^{n}\right\}$; $\frac{3}{8}$;
$\frac{4}{21}\left\{1-\overline{\frac{1}{8}}\Big|^{n}\right\}$; $\frac{4}{21}$; $\frac{1}{12}\left\{1-\left(-\frac{1}{4}\right)^{n}\right\}$; $\frac{1}{12}$.

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## LXIX.—Pp. 220, 221.

(1.) $2^{n+2}+n-4$; $n$; $\frac{n(n+1)}{2}+2^{n+2}-4$;
$\frac{1}{2}\left(1-\overline{\frac{1}{2}}\Big|^{n}\right)+\frac{n(n+3)}{8}$; $a\left(\frac{3^{n}}{2}-\frac{1}{2}-n\,\overline{n+1}\right)$.
(2.) $\frac{1}{6}n(n+1)(2n+1)$; $\frac{n}{3}(4n^{2}+12n+11)$; 11036; 2660;
$n(441n^{2}+315n+26)$. (3.) $na^{3}+\frac{nab^{2}}{2}(2n^{2}-3n+1)-$
$\frac{3}{2}a^{2}b^{2}(n^{2}-n)+\frac{n^{2}b^{3}}{4}(n^{2}-2n+1)$. The next series is equivalent to $2^{3}+3^{3}+$ &c. to $n$ terms, $+\,6(2^{2}+3^{2}+$ &c. to $n$ terms$)\,+$ $8(2+3+$ &c. to $n$ terms$)$.

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## Ex. LXX.—P. 227.

(1.) 720. (2.) 99720. (3.) $\frac{\underline{|12}}{\underline{|2}\;\underline{|2}\;\underline{|3}}$. (4.)
(5.) $\underline{|8}$, $(\underline{|4})^{2}$. (6.) 576. (7.) 720. (8.) 542640. (9.) 2100.
(10.) 25 and 20. (11.) $n(n-1)(n-2)$ and 120. (12.) 70.
(13). $\frac{1000\cdot 999,\ \text{\&c. to 50 terms}}{1\cdot 2,\ \text{\&c.}\quad ,,}$ and $\frac{999\cdot 998,\ \text{\&c. to 49 terms}}{1\cdot 2,\ \text{\&c.}\quad ,,}$.
(14.) 19. (23.) 6.

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### Ex. LXXI.—P. 234.

(1.) $a^7 + 7a^6x \; \ldots\ldots + x^7$; $1 + 45b + 900b^2$, &c.; $256a^{16} - 3072a^{14}x^2$; $x^{24} - 12x^{22}y^2 + 66x^{20}y^4 -$, &c.; $a^{13} - \frac{13}{4}\,a^{12}b + \frac{39}{4}\,a^{11}b^2 -$, &c.; $\left(\frac{3a}{5}\right)^6 - 6\left(\frac{3a}{5}\right)^5 + 15\left(\frac{3a}{5}\right)^4 -$, &c.; $\left(\frac{ab}{2}\right)^{11} - 11\left(\frac{ab}{2}\right)^9 + 55 \left(\frac{ab}{2}\right)^8 -$, &c.; $a^{\frac{n+1}{2}} - (n+1)\,a^{\frac{n}{2}}x^{\frac{1}{2}} + \frac{(n+1)\,n}{2}\,a^{\frac{n-1}{2}}\,x -$, &c.; $1 - 3nx + \frac{3n\,(2n-1)}{4}x^2 -$, &c.

(2.) $220x^3$; $\frac{p\,(p-1)\,(p-2)\,(p-3)}{24}\,a^{2p-8}\,b^8$; $\frac{(n+3)\ldots\ldots(n)}{24}\,x^4\,a^{n-1}$; $\frac{(2p+1)\;\ldots\ldots\;(p+2)}{1.2\;\ldots\ldots\;p}\,x^{p+1}y^p$; $\frac{1.3.5\ldots\ldots(2n-1)}{|\underline{n}}\,2^n$; $\frac{|\underline{4p}}{(|\underline{2p})^2}\,.\,a^{4p}$.

(3.) $13$; $\frac{2p\;\ldots\ldots(2p-7)}{|\underline{8}}\,a^{2p-8}$; $\frac{p\,(p-1)\ldots\ldots\left(\frac{p}{2}+2\right)}{\left|\underline{\frac{p}{2}-1}\right.}\;\frac{1}{a^2}$; the coefficient of $x^{\overline{r-n-4}}$ in $(x-2a)^{n+4}$.

(4.) $\frac{24.\ldots\ldots18}{|\underline{7}}\;3^{17}\,2^7$; $\frac{(n-2)\,(n-3)\,(n-4)}{6}$, $\frac{1.3.\ldots\ldots(4n-1)}{|\underline{2n}}.2$, $495c^8$

(5.) $91$; $\frac{(n+1)\,(n+2)}{2}$. (6.) 35; 209352. (7.) $2^{14} - 2$. (8.) 1584.

(9.) $r - 1$. (10.) $n = 9$ or $-2$. (11.) $\frac{2^{n+1}-1}{n+1}$.

(14.) $\frac{|\underline{4n}}{(|\underline{2n})^2}$. (15). $\frac{4}{3}\,\frac{n-2}{n-3} = \frac{b^2}{ac}$.

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### Ex. LXXIII.—P. 250.

(1.) 10; 3628800; − 12; 9; − 728.

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### Ex. LXXVII.—P. 280.

(1.) £15 9*s*. 4½*d*.; £1043 10*s*. 11¼*d*. (2.) £543 9*s*. 6½*d*.; £358 10*s*. 10¾*d*. (6.) £1688 10*s*. 5*d*.; £260 13*s*. 11½*d*.

£1464 4*s*. 10¾*d*. (7.) £101 12*s*. 7*d*. (8.) 5·07. (9.) £100.

(10.) $\log.(1+r) = \frac{1}{n}\log. t$. (11.) 4½ per cent. (12.) £46 9*s*.

(13.) £2050; £2785 19*s*. 2½*d*.; £2821 0*s*. 1*d*. (14.) 10$\frac{1}{20}$.

(15.) $P\{\overline{1+r'}|^{s} - \overline{1+r}|^{a}\}$. (16.) £6226 3*s*. 7*d*.

(17.) $a\left(1+\frac{r}{100}\right)^{n} = \frac{100b\left\{\overline{1+\frac{r}{100}}\right|^{n} - 1\Big\}}{r}$.

(18.) $\frac{a(1\cdot04)^{b}}{25\{\overline{1\cdot04}|^{b} - 1\}}$. (19.) 121 nearly. (21.) 6 nearly.

(22). $\frac{\left(1+\frac{r}{12}\right)^{20}}{20} = (1+r)^{n}$. (23.) 3.

(24.) $\frac{3(1+r)^{30} - 3}{r(1+r)} = 100$. (25.) 225. (26.) 4⅔.

(27.) £405 2*s*. (28.) 1930 16*s*. 7*d*. (29.) £4077 1*s*. 10*d*.

(30.) £1773 7*s*. (32.) 5¼. (33.) $\frac{a\overline{1+2}|^{-p} - a\overline{1+2}|^{\overline{p+q}}}{r}$.

(35.) £62 10*s*. at 5 per cent. (36.) £6139 2*s*. (38.) 5⅛.

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## Ex. LXXVIII.—P. 265.

(1.) $\frac{n}{4}(1006 + 493n + 234n^2 + 27n^3)$; $\frac{4n}{3}(n^2 + 6n + 11)$.

(2.) $\frac{1}{42} - \frac{1}{6(6n+7)}$; $\frac{1}{1040} - \frac{1}{10(5n+8)(5n+13)}$.

(3.) These are found by the formula III. in page 264.

(4.) These are found by the formula IV. in page 265.

(5.) These are found by the formula III. and IV. in pages 264 and 265.

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## Ex. LXXIX.—P. 269.

(1.) 7140. (2.) 22100 and 59640. (3.) 364; 816; 1140. (4.) 6896. (5.) 19800. (6.) 18731. (7.) 20. (8.) 18. (9.) 190 and 45. (10.) 210. (11.) 38024. (12.) 338350. (13) 9455. (14.) 3311. (15.) 21. (16.) 12685. (17.) 1824. (18.) 60 balls in each side of base. (19.) 400. (20.) 385 and 220.

(21.) 2584. (22.) 20720; 14105. (23.) 12470. (24.) 15625.
(25.) 9095. (26.) 15435. (27.) 23.

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## Ex. LXXX.—P. 273.

(1.) 41; 13284; 4428; 123; 7000005; 466667; 1·476; ·0246; $\frac{3}{4}$; $\frac{17}{43}$; $\frac{15}{13}$. (2.) 1·41421; ·04472; 3·67424; 1·42302; ·000836; 5·3796; ·4472; ·4714: 1·80277; ·74554; ·15242; 2·38047.

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## Ex. LXXXI.—P. 276.

(1.) 92; 39; 13; 276; 12636; 4416. (2.) 1·17; 5·52; ·2208; ·00351; 1·104. (3.) 105·3; ·13248; ·6318.

BIBLIOTHECA BODLEIANA
DOMINUS ILLUMINATIO MEA

LONDON: PRINTED BY WILLIAM CLOWES AND SONS, STAMFORD STREET,
AND CHARING CROSS.